DAHLGREN DIVISION
NAVAL SURFACE WARFARE CENTER
Dahlgren, Virginia 22448-5100

NSWCDD/MP-95/87

FUNDAMENTALS AND FOUNDATIONS OF MOBILE LONG-RANGE BALLISTIC MISSILES

BY KEE SOON CHUN

STRATEGIC AND SPACE SYSTEMS DEPARTMENT

APRIL 1996

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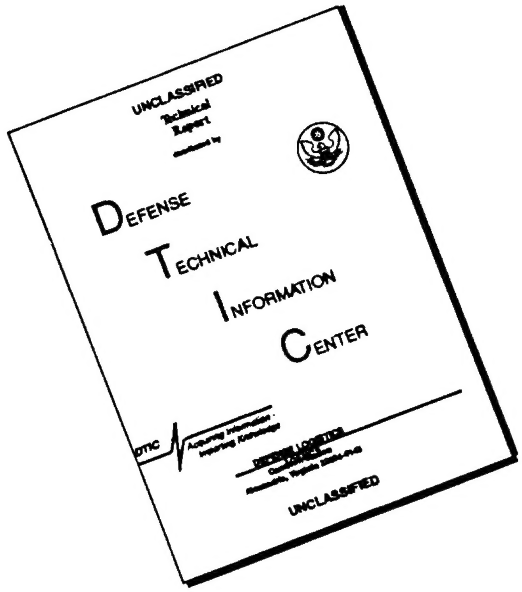

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REPORT DOCUMENTATION PAGE

Form Approved
OBM No. 0704-0188

1. AGENCY USE ONLY (*Leave blank*)

2. REPORT DATE
April 1996

3. REPORT TYPE AND DATES COVERED

4. TITLE AND SUBTITLE
Fundamentals and Foundations of Mobile Long-Range Ballistic Missiles

5. FUNDING NUMBERS

6. AUTHOR(s)
Kee Soon Chun

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Surface Warfare Center
Dahlgren Division (Code K41)
17320 Dahlgren Rd.
Dahlgren, VA 22448-5100

8. PERFORMING ORGANIZATION REPORT NUMBER
NSWCDD/MP-95/87

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (*Maximum 200 words*)

This publication is a training guide explaining the fundamentals and foundations of the long-range ballistic missile weapon systems launched from moving bases such as submarine or aircraft. Since the land-based missiles are essentially no different from the missiles launched from the moving bases except that the bases are fixed on the earth (without rotational and linear motions), the theory and analyses explained in this training guide would be equally applicable to all three classes of missiles—land-based, sea/undersea-based, or airborne missiles, which use inertial navigation system.

This publication is written mainly for the training of new technical members using only introductory physics and calculus. Nonetheless, it manages to maintain the robustness of theory and the rigor of mathematical derivation. For this reason, it is recommended to veteran technical members who might find a segment or two that would shed a new light on old subjects.

14. SUBJECT TERMS
Long-Range Ballistic Missile (LRBM)
Newton's Second Law
Theorem of Coriolis
Pendulous Integrating Gyro Accelerometer

15. NUMBER OF PAGES
127

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

NSN 7540-01-280-5500

Standard Form 298 (Rev 2-89)
Prescribed by ANSI std. Z39-18
298-102

FOREWORD

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It is written mainly for the training of new technical members using only introductory physics and calculus. Nonetheless, it manages to maintain the robustness of theory and the rigor of mathematical derivation. For this reason, it is recommended to veteran technical members as well, who might find a segment or two that would shed a new light on old subjects.

On behalf of the author, Dr. Chun, the appreciations are expressed to the following people: to Mark Hall for reviewing the entire manuscripts from the viewpoint of theory and analysis; to Gerald Stoehr, Barry Bressler, Davis Owen, Alan Rufty, J. C. Brown, Gary Sitzman, and Linda Chisholm for comments and discussions; to Patrick Godin for grammatical and typographical corrections; to Richard Hearn for typing unwieldly symbols; to W. H. Horton for his supervisory encouragement along with useful comments; and finally to author's teacher and advisor, Professor Walter Hollister of MIT Department of Aeronautics and Astronautics for allowing him to use some figures on gyros from his book (Reference 1).

This report was reviewed by William H. Horton, Head, Systems and Environment Branch of the SLBM Research and Analysis Division.

Approved by:

DAVID B. COLBY, Head
Strategic and Space Systems Department

CONTENTS

CONTENTS (CONTINUED)

ILLUSTRATIONS

ILLUSTRATIONS (CONTINUED)

ILLUSTRATIONS (CONTINUED)

1.0 INTRODUCTION

This publication is a training guide explaining fundamentals and foundations of the long-range ballistic missiles (LRBM) from the viewpoint of the navigation and guidance. The missiles may be launched from either fixed-base on land or from moving-base such as aircraft or submarine. The articles are written at such a level that anyone with freshman physics and calculus would not have much difficulty in understanding them. Readers of this publication are cautioned that the analysis of the actual, real-world systems are much more complicated than those described here. As the title implies, these articles are merely "fundamentals and foundations."

Section 2.0 lists the symbols and notations used in this report. It includes an operator $P(\cdot)$ for $\frac{d}{dt}(\cdot)$. This is handy because the differentiation with respect to time in the earth-fixed E-frame may simply be expressed as $P_E(\cdot)$. Section 3.0 describes the frames of reference relevant to this report. Section 4.0 explains implementation of Newton's second law of motion for the purpose of inertial navigation. Since all LRBMs use the center of the earth as the origin of the inertial reference frame, Section 5.0 deals with the errors incurred in the earth-centered inertial frame. Section 6.0 explains that the effect of angular rotations are dependent on the sequence in which they are executed, unlike vector additions. Theory of inertial navigation involves rotation of one reference frame relative to another. The theorem of Coriolis, which is discussed in Section 7.0, is essential to understand the relationship between the two rotating frames relative to each other. The matrix form of the theorem of Coriolis is discussed in Section 8.0. Section 9.0 discusses the rotational form of Newton's second law of motion, which is essential for the understanding of the operation of gyros, gyro-accelerometers, and stable platforms of guidance system of LRBMs. Section 10.0 describes the prototype linear accelerometer, which serves as a tutorial for understanding the pulse integrated pendulous accelerometers (PIPA), and pendulous integrating gyro accelerometers (PIGAs), is described in Section 11.0. Section 12.0 discusses the rotational motion of the missile body based on some simple assumptions, which allows us to avoid complicated mathematics.

Based on theoretical and mathematical tools developed in previous sections, the theory of practical gyros are explained in Section 13.0, the single degree-of-freedom (SDOF) gyro in Section 14.0, the stable platform of inertial measurement unit in Section 15.0, and the PIGA in Section 16.0.

Moving from components to the system level, the derivation and implementation of the equation of motion for the inertial navigation is described in Section 17.0. The determination of the initial position and initial velocity to be used in the integrations by the guidance computer of the Newton's second law of motion is described in Section 18.0. Moving to the rocket steering scheme during powered stages, the so-called cross product steering is described in Section 19.0.

Toward the end of the powered flight, the guidance system of the missile tries to correct its error, in a statistical sense, by viewing a predesignated star. The weighing matrix, also called the W matrix, which is used in the statistical computation for this purpose, is derived in Section 20.0. The topic of Schuler tuning is discussed concisely in Section 21.0. Schuler tuning is conceptually

important because it deals with the prevention of platform misorientation caused by the acceleration in the earth's gravitational field. Finally, the correlated velocity, which is the required velocity of the reentry vehicle to hit the target on the rotating earth by free falling with specified time-of-flight (TOF), is discussed in Section 22.0.

2.0 LIST OF SYMBOLS AND NOTATIONS

R_{PQ} Displacement vector from point P to point Q. Similarly V_{PQ} is the velocity vector of Q relative to P, etc.

W_{AB} Angular velocity of Frame B relative to Frame A.

V^A Any vector, (e.g., R or W, etc.) with components expressed in Frame A.

$$P(\cdot) = \frac{d}{dt}(\cdot)$$

$$\frac{1}{P}(\cdot) = \int (\cdot)\, dt$$

$$P^2(\cdot) = \frac{d^2}{dt^2}(\cdot)$$

$P_A(\cdot) \equiv \frac{d_A}{dt}(\cdot)$ Time derivative with respect to (WRT) Frame A, (i.e., the increment observed in Frame A).

Thus,

$$P_A R = P(R^A) = \frac{d}{dt}(R^A) = \frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix}^A$$

$$P_A^2 R = P_A(P_A R) = P(PR^A) = \frac{d^2}{dt^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}^A$$

$$P_A P_B R = P_A(P_B R)$$

The advantages of these notations will become obvious in the second-half of this report.

Let

$$R^D = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^D \quad \text{and} \quad W_{AB}^D = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}^D ,$$

then the matrix equivalent of the vector W_{AB}^D, denoted by W_{AB}^{DK} is given by

$$W_{AB}^{DK} = \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix}$$

so that $\left[W_{AB}^{DK}\right] R \leftrightarrow W_{AB}^D \times R^D$ or

$$\begin{pmatrix} 0 & -w_Z & w_Y \\ w_Z & 0 & -w_X \\ -w_Y & w_X & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{vmatrix} i & j & k \\ w_X & w_Y & w_Z \\ x & y & z \end{vmatrix}_D \Leftrightarrow \begin{pmatrix} w_y z - w_z y \\ w_z x - w_x z \\ w_x y - w_y x \end{pmatrix}$$

C_A^B coordinate transformation matrix from Frame A to Frame B.

Note:

$$C_B^N C_I^B = C_I^N$$

$C_B^A C_A^B = C_A^A = I$ identity matrix.

3.0 FRAMES OF REFERENCE

3.1 INERTIAL (I)-FRAME

The frame in which Newton's laws of motion (specifically the second law) is valid is called inertial (I)-frame by definition. In all, LRBM, an earth-centered nonrotating frame in the inertial space, is accepted as a (quasi) I-frame. Obviously, since the center of the earth accelerates around the sun in an elliptic orbit, the earth-centered I-frame (denoted by I_E) is not an I-frame in a strict sense. However, the error incurred in this practice is negligible (this is shown in detail under a separate heading in the sequel).

In the standard I_E-frame, the z-axis is collinear with the earth's polar (spin) axis. The x-axis and y-axis are on the earth's equatorial plane (but not rotating with the earth) in such a way as to form an orthogonal triad by the right-hand rule. This frame is sometimes referred to as the "polar earth-centered inertial frame."

3.2 EARTH (E)-CENTERED, EARTH-FIXED FRAME

This frame, being earth fixed, rotates with the earth relative to the inertial space. The x, y, and z axes are defined similar to the standard I_E frame, except x and y axes are fixed to the earth and therefore rotate with the earth.

3.3 NAVIGATIONAL (N), LOCAL-LEVEL FRAME

The origin of this frame is located at the center of the onboard inertial navigation system (INS) of the ship, submarine, or aircraft cruising on or near the surface of the earth. The x-axis points to the north and y-axis to the east, both on the local-level plane, which is tangent to the reference ellipsoid. The z-axis points downward perpendicular to the tangential plane. The z-axis may or may not coincide with the local vertical (plumb bob) lim because the earth's gravity field is not uniform. This causes the deflection of the vertical.

3.4 GUIDANCE (G)-SYSTEM INERTIAL MEASUREMENT UNIT (IMU) FRAME

The origin of the G-frame is located at the center of the missile guidance system's IMU, with its axes parallel to the earth-centered I-frame. Obviously, since the missile is subject to acceleration, the G-frame cannot be an I-frame. However, since its axes are parallel to the axes of the I-frame, the coordinate transformation matrix from the G-frame to the I-frame denoted by C_G^I must be an identity matrix or $C_G^I = I$, where I denotes the identity matrix.

The onboard accelerometer measures the acceleration of the instrument relative to the origin of the I-frame, not relative to the origin of the G-frame to which the instrument has a fixed distance and therefore the relative acceleration should be zero. However, since the axes of G-frame is parallel to the axes of I-frame, the accelerometer output (sensed acceleration and the velocity-gain caused by it during the computation interval) should have the same values whether they are expressed in I-frame or G-frame. That is,

$$\Delta V^I = C_G^I \Delta V^G = I \Delta V^G = \Delta V^G$$

where

ΔV^G = velocity gain (ΔV) determined by the guidance system with components expressed in G-frame, and

$\Delta V^I = \Delta V$ with components expressed in I-frame.

Thus

$$\Delta R^I = \Delta V^I \Delta t = \Delta V^G \Delta t$$

where

ΔR^I = position displacement with components expressed in the I-frame

We see that as far as the determination of the missile's incremental position and velocity are concerned the G-frame behaves as if it were an I-frame, although in reality it is not. To determine the current position or velocity relative to the I-frame, the incremental position or velocity obtained in the guidance frame is added by the guidance computer to the previous position or velocity, referenced to the origin of the I-frame, which is conventionally the center of the earth in all LRBM applications. For this reason, some incorrectly call the G-frame "inertial," which should be avoided.

4.0 NEWTON'S SECOND LAW (FOR THE IMPLEMENTATION OF INS)

According to Newton's second law

$$mP_I^2R_{IP} = \Sigma F \tag{4-1}$$

Since the sum of externally applied forces ΣF has to be the sum of the gravitational force F_G and the nongravitational force F_{NG}, we may write (1) as follows:

$$P_I^2R_{IP} = \frac{F_G}{m} + \frac{F_{NG}}{m} = g + f \tag{4-2}$$

where g is the gravitational force per unit mass and f is the nongravitational force per unit mass.

By convention, f is called simply "specific force," meaning specific (per unit mass) nongravitational force.

The reason we separate $f = \frac{F_{NG}}{m}$ from $g = \frac{F_G}{m}$ is by necessity, not by choice, because the on-board inertial instrument (accelerometer) can measure only f, not g. This is explained in the section of the accelerometer later. The g is determined by the onboard computer, based on the mathematical model (inverse square law) as a function of the position R_{IP} from the center of the earth, which is the origin of the I-frame by convention.

Thus, in the INS, Newton's second law (4-2) is implemented as shown in Figure 4-1.

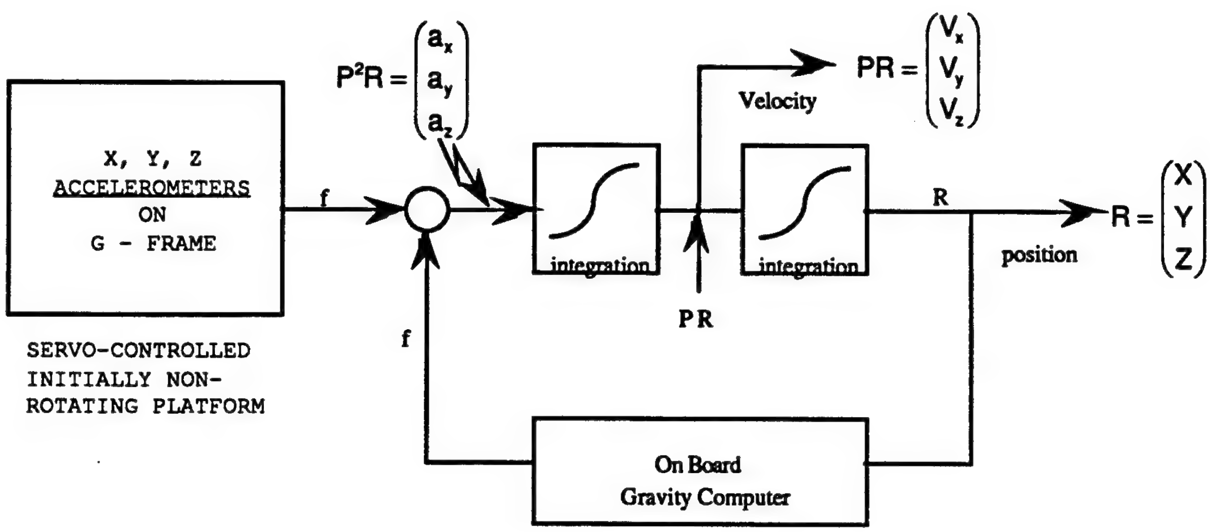

FIGURE 4-1. NEWTON'S SECOND LAW IMPLEMENTATION OF INS

5.0 ERRORS IN THE EARTH-CENTERED INERTIAL FRAME (I_E FRAME)

For the purpose of estimating the magnitude of the errors incurred by treating the earth-centered inertially nonrotating frame (I_E frame) as the I-frame, consider a system consisting of the sun (S), the earth (E), the moon (M), and a missile (P) near the earth with the center of the sun considered as the origin of an I-frame as shown in Figure 5-1 (Figure not to scale; the rest of the universe is ignored).

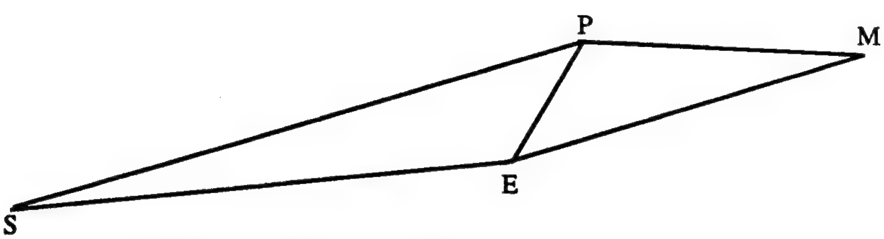

FIGURE 5-1. THE SUN, MOON, AND EARTH SYSTEMS

The forces acting on missile P with mass m are the gravitational forces by the sun, the earth, the moon, and the nongravitational forces of thrust and aerodynamic force (if in the atmosphere).

By applying Newton's second law along with the law of gravitation to the missile at P.

$$-\frac{GM_s m}{R_{SP}^2}\underline{U}_{SP} - \frac{GM_E m}{R_{EP}^2}\underline{U}_{EP} - \frac{GM_m m}{R_{mp}^2}\underline{U}_{MP} + F_{NG} = mP_I^2\underline{R}_{SP} \tag{5-1}$$

m = mass of the missile at P

M_s = mass of the sun at S

M_E = mass of the earth at E

M_M = mass of the moon at M

$\underline{U}_{SP}$ = unit vector along $\underline{SP}$

$\underline{U}_{EP}$ = unit vector along $\underline{EP}$

$\underline{U}_{MP}$ = unit vector along $\underline{MP}$

G = gravitational constant

Since $\underline{R}_{SP} = \underline{R}_{SE} + \underline{R}_{EP}$

$$mP_I^2 R_{SP} = mP_I^2 \underline{R}_{SE} + mP_I^2 \underline{R}_{EP} \tag{5-2}$$

Dividing by (1), by m and using (2):

$$-\frac{GM_S}{R_{SP}^2}\underline{U}_{SP} - \frac{GM_E}{R_{EP}^2}\underline{U}_{EP} - \frac{GM_M}{R_{mp}^2}\underline{U}_{MP} + \frac{F_{NG}}{m} = P_I^2 R_{SE} + P_I^2 R_{EP}\,. \tag{5-3}$$

Now, applying Newton's second law to the earth:

$$M_E P_I^2 R_{SE} = -\frac{GM_S M_E}{R_{SE}^2}\underline{U}_{SE} - \frac{GM_M M_E}{R_{ME}^2}\underline{U}_{ME} \tag{5-4}$$

or

$$P_I^2 R_{SE} = -\frac{GM_S}{R_{SE}^2}\underline{U}_{SE} - \frac{GM_M}{R_{ME}^2}\underline{U}_{ME}\,. \tag{5-5}$$

Substituting (5) into (3) for $P_I^2 R_{SE}$

$$P_I^2 R_{EP} = -\frac{GM_E}{R_{EP}^2}\underline{U}_{EP} + \frac{F_{NG}}{m} + \Delta g_S + \Delta g_M \tag{5-6}$$

where

$$\Delta g_S = -GM_S\left(\frac{1}{R_{SP}^2}\underline{U}_{SP} - \frac{1}{R_{SE}^2}U_{SE}\right) \tag{5-7}$$

is the difference between gravitational forces per unit mass by the sun on the missile and by the sun on the earth, and

$$\Delta g_M = -GM_M\left(\frac{1}{R_{MP}^2}\underline{U}_{MP} - \frac{1}{R_{ME}^2}\underline{U}_{ME}\right) \tag{5-8}$$

is the difference between gravitional forces per unit mass by the moon on the missile and that by the moon on the earth.

For a missile about 1,000 km above the surface of the earth:

$\frac{\Delta g_s}{g_E}$ is less than one part per 10 millions in magnitude.

$\frac{\Delta g_M}{g_E}$ is less than two parts per 10 millions in magnitude,

where $g_E = \frac{GM_E}{R_{EP}^2}$ is the magnitude of the earth's gravitational force on the missile per unit mass.

These values are at least one order of magnitude smaller than the accuracy of the best state-of-the-art accelerometer even in the laboratory environment.

Although we have not considered the effect of the rest of the universe on the missile, it turns out that its magnitude is much smaller than the effects from the sun and moon.

For these reasons, (6) is approximated by

$$P_I^2 R_{EP} = -\frac{GM_E}{R_{EP}^2} \underline{U}_{EP} + \frac{F_{NG}}{m} \tag{9}$$

with negligible error represented by (7) and (8).

Equation (9) is the exact form of Newton's second law $\Sigma F = ma$ or $a = \frac{\Sigma F}{m}$ as if the origin of the I-frame were located at the center of the earth.

Qualitatively speaking, the magnitude of Δg_s as a proportion in $P_I^2 R_{EP}$ is neglible because the sun pulls both the earth and missile together so that the relative position of the missile with respect to the earth is relatively unaffected. Similar reasoning applies to the case of Δg_m being negligible.

6.0 ANGULAR ROTATIONS

In the addition of two vectors V_1 and V_2, $V_1 + V_2 = V_2 + V_1$ as shown in Figure 6-1, (i.e., V_1 followed by V_2 has the same result as V_2 followed by V_1).

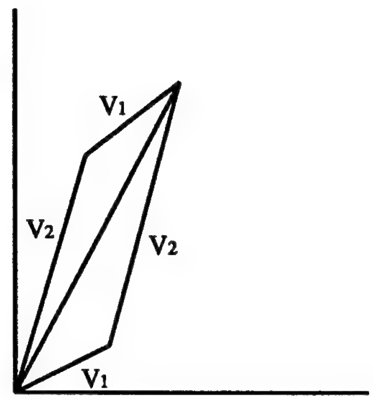

FIGURE 6-1. VECTOR ADDITION

In angular rotations, a rotation about the x-axes followed by another rotation about the displaced y-axes does not result in the same orientation of the frame as a rotation about the y-axes followed by another rotation about the displaced x-axes. In this sense, angular rotations do not commute, (i.e., do not act like vectors).

To see this by way of two simple demonstrations, consider an orthogonal (Frame-0) with X_0, Y_0, and Z_0 axes, as shown in Figure 6-2.

<u>Frame 0</u>

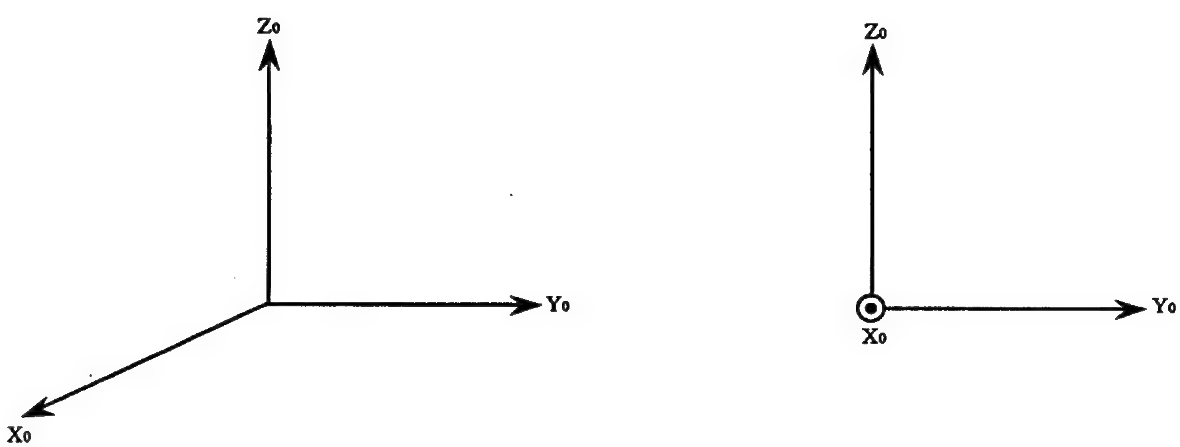

FIGURE 6-2. O-FRAME

In the first demonstration, rotate the above frame about X_0 - axes by 90° (using the right-hand rule). This results in Frame 1 (with X_1, Y_1, and Z_1 axes) as shown in Figure 6-3.

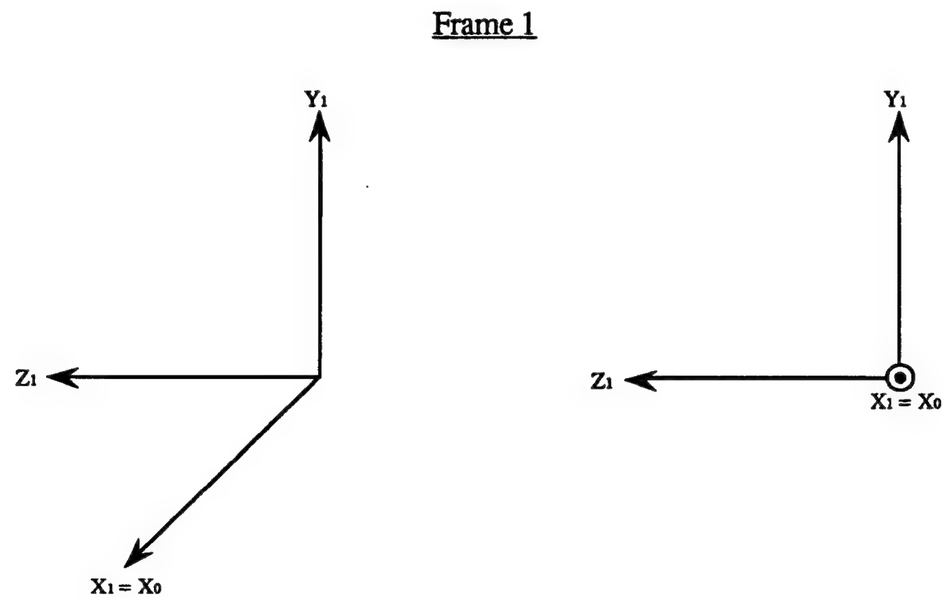

FIGURE 6-3. FRAME 1

Next, rotate Frame-1 about Y_1 - axes by 90°—resulting in Frame-2 (with X_2, Y_2, Z_2 axes) as shown in Figure 6-4.

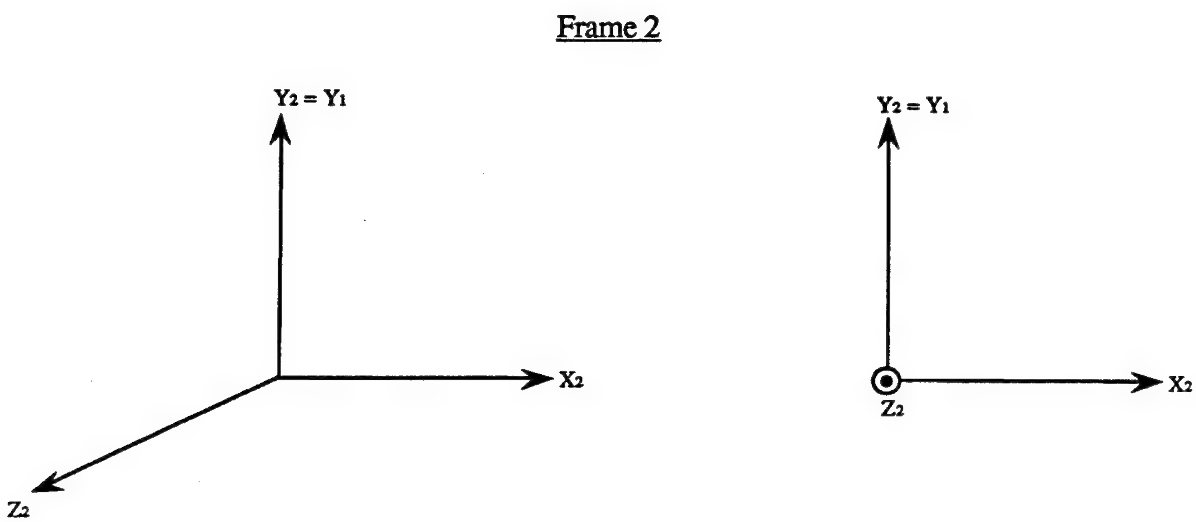

FIGURE 6-4. FRAME 2

Now in the second demonstration, returning to Figure 6-2, rotate Frame 0 about Y_0-axes (instead of X_0 axes) by 90°—resulting in Frame 1′ (with X_1', Y_1', and Z_1' axes) as shown in Figure 6-5.

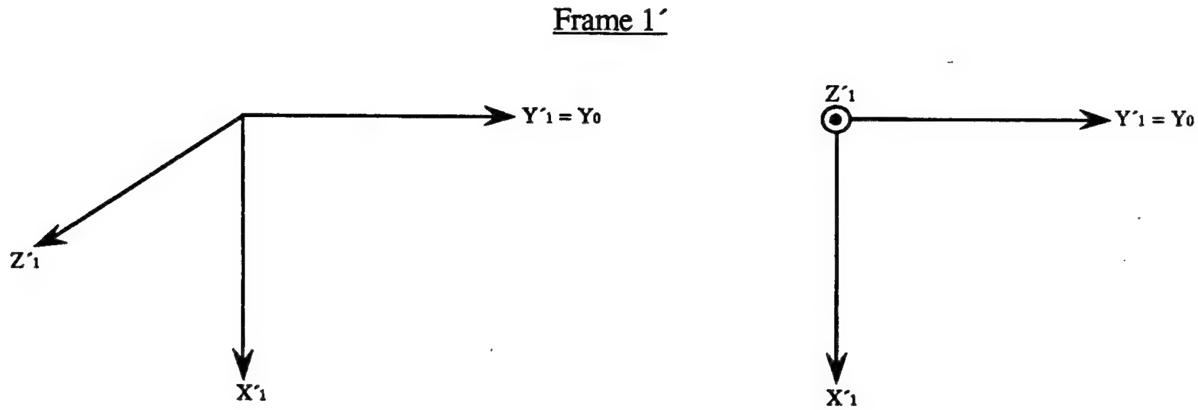

FIGURE 6-5. FRAME 1´

Next, rotate Frame 1´ about X_1'-axes by 90°—resulting in Frame 2´ (with X_2', Y_2', and Z_2' axes) as shown in Figure 6-6.

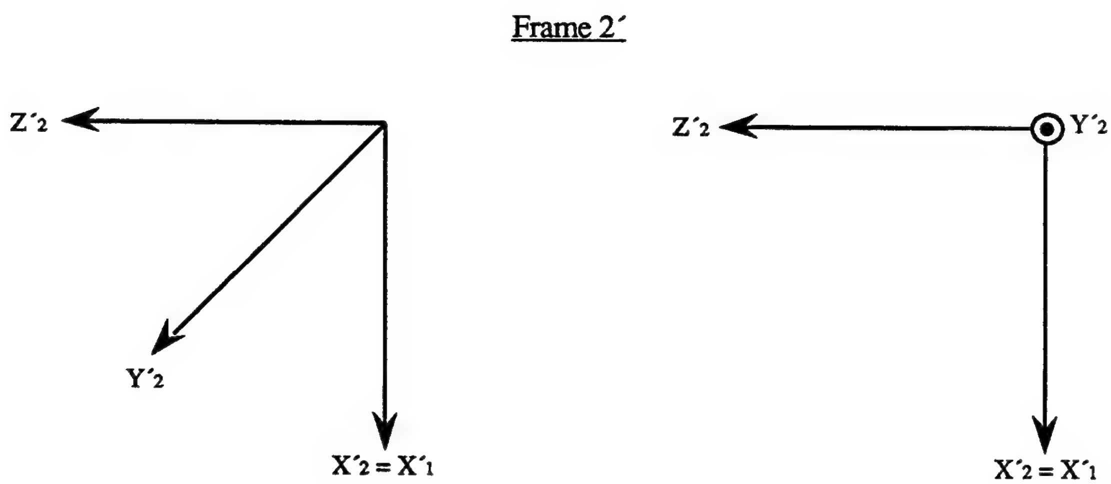

FIGURE 6-6. FRAME 2´

Obviously, Frame 2´ (Figure 6-6) has a different orientation from Frame 2 (Figure 6-4). It means that, in describing the effects of angular rotations, it is necessary to specify the order in which a sequence of rotations is executed.

Consider Frames A, B, C, and a vector R, which has components $R^A = (X_A, Y_A, Z_A)^T$ in Frame A, $R^B = (X_B, Y_B, Z_B)^T$ in Frame B, and $R^C = (X_C, Y_C, Z_C)^T$ in Frame C.

Then

$$R^B = C_A^B R^A \tag{6-1}$$

and

$$R^C = C_B^C R^B \tag{6-2}$$

Substituting (6-1) into (6-2) for R^B

$$R^c = C_B^C C_A^B R^A \tag{6-3}$$

Since

$$R^C = C_A^C R^A . \tag{6-4}$$

From (6-3) and (6-4), we conclude

$$C_B^C C_A^B = C_A^C . \tag{6-5}$$

Referring to (6-5) and based on our discussions, $C_B^C C_A^B \neq C_A^B C_B^C$

As a matter of fact, $C_A^B C_B^C$ has neither physical nor geometrical meaning. In our notation, the second rotation C_B^C is written to the left of the first rotation C_A^B so that the superscript B of C_A^B matches the subscript B of C_B^C, so that $C_B^C C_A^B$ becomes C_A^C as in (6-5).

We can extend this process, for instance, to:

$$C_D^E C_C^D C_B^C C_A^B = C_A^E \tag{6-6}$$

By writing as in (6-5) or (6-6), we can check the consistency of multiplications of the coordinate transformation matrices. If superscripts and subscripts do not match in a way we have specified, it indicates errors somewhere.

7.0 THEOREM OF CORIOLIS

Consider two references, Frames A and B, with common origin at 0. Frame B rotates with respect to Frame A with angular velocity W_{AB}. A vector $R_{op} = R$ may be expressed in Frame A

$$R^A = IX + JY + KZ \tag{7-1}$$

and in Frame B

$$R^B = ix + jy + kz\,. \tag{7-2}$$

Now, using (7-1) :

$$\begin{aligned} P_A R^A &= I\frac{dX}{dt} + J\frac{dY}{dt} + K\frac{dZ}{dt} \\ &\quad + X\frac{dI}{dt} + Y\frac{dJ}{dt} + Z\frac{dK}{dt} \\ &= I\frac{dX}{dt} + J\frac{dY}{dt} + K\frac{dZ}{dt} \end{aligned} \tag{7-3}$$

because the unit vectors I, J, and K are fixed in Frame A, and do not vary with time in Frame A.

Using (7-2):

$$P_A R^B = i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt} + x\frac{di}{dt} + y\frac{dj}{dt} + z\frac{dk}{dt}\,. \tag{7-4}$$

Note: In this case, the unit vectors i, j, and k, which are fixed in Frame B, rotate relative to Frame A, and therefore, are time-varying as viewed from Frame A.

Now, consider $\frac{di}{dt}$ in (7-4). (Referring to Figure 7-1).

Since vector i is a unit vector, it cannot change its magnitude. However, it can change its direction relative to Frame A because i is fixed in Frame B, which rotates relative to Frame A. For an infinitesimal angular displacement, we can visualize the tip of the i-vector moving on the plane parallel to the j - k plane. Therefore, we may express Δi caused by the angular displacement of i in Frame A in terms of its displacements in the j and k directions.

Referring to the Figure 7-1 below, we see

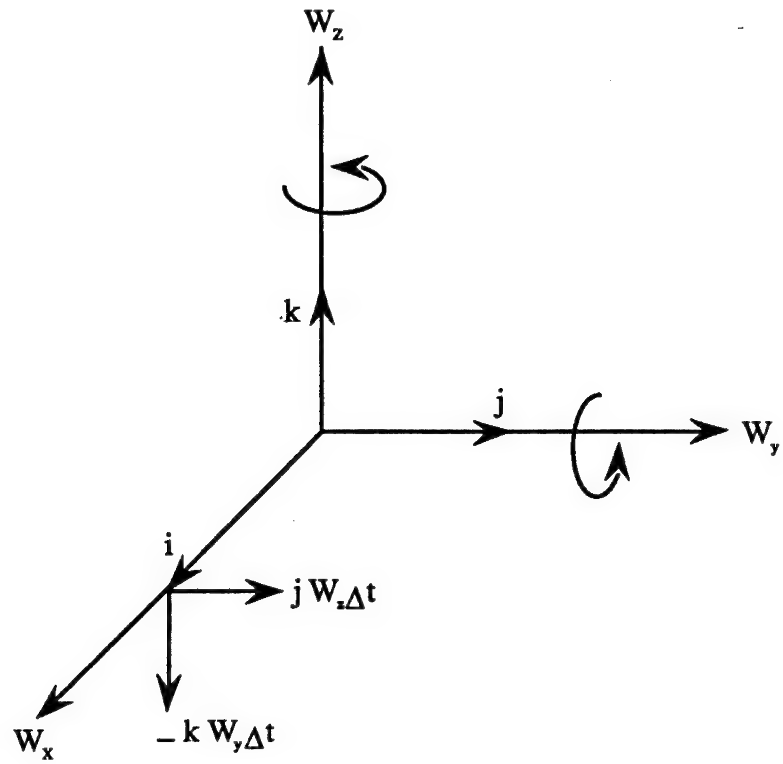

FIGURE 7-1. INCREMENTAL ROTATION OF FRAME

that W_Z causes $\Delta i = jW_Z\Delta t$ in the j direction during Δt and W_y causes $\Delta i = -kW_y\Delta t$ during Δt in the $-k$ direction. Summing the two components, we have

$$\Delta i = jW_Z\Delta t - kW_y\Delta t \tag{7-5}$$

dividing by Δt and taking the limit, we have

$$\frac{di}{dt} = jW_z - kW_y. \tag{7-6}$$

It turns out that the right-hand side of (7-6) is also equal to $W^B \times i$ as shown below:

$$W^B \times i = \begin{vmatrix} i & j & k \\ W_x & W_y & W_z \\ 1 & 0 & 0 \end{vmatrix}$$

$$= jW_Z - kW_y \tag{7-7}$$

It follows:

$$\frac{di}{dt} = w^B \times i . \tag{7-8}$$

Similarly, we can show that:

$$\frac{dj}{dt} = w^B \times j \tag{7-9}$$

$$\frac{dk}{dt} = w^B \times k . \tag{7-10}$$

Now, referring to the right-hand side of (7-4), and using (7-8), (7-9), and (7-10):

$$x\frac{di}{dt} + y\frac{dj}{dt} + z\frac{dk}{dt}$$

$$= x\left(w^B \times i\right) + y\left(w^B \times j\right) + z\left(w^B \times k\right)$$

$$= w^B \times (ix + jy + kz)$$

$$= w^B \times ix + w^B \times jy + w_B \times kz \quad \text{(since x, y, and z are scalars)}$$

$$= w^B \times R^B$$

$$= w^B_{AB} \times R^B \quad \text{(since } W = W_{AB} \text{ by definition in our case)} \tag{11}$$

Now,

$$P_B R^B = P\,(ix + jy + kz)^B$$

$$= i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt} \tag{12}$$

because i, j, k are fixed in Frame B.

So, substituting (7-12) and (7-11) into (7-4):

$$P_A R^B = P_B R^B + W^B_{AB} \times R^B . \tag{13}$$

In (13), each term may be expressed, after appropriate transformations, in either Frame A or Frame B, (see next section). So, dropping the superscript B from (7-13),

$$P_A R = P_B R + W_{AB} \times R \tag{7-14}$$

which is called the *Equation of Coriolis*.

What the Equation of Coriolis implies is that the differentiation (WRT time) of a vector in one frame is not equal to the differentiation of the same vector in another frame, which is rotating with respect to the first frame. And to obtain the value of $P_A R$ of (7-3) in terms of $P_B R$ of (7-12), we have to add a correction term W_{AB} x R (which incorporates the effect of rotation of Frame B relative to Frame A) to $P_B R$ as shown in (7-14).

For some, the Equation of Coriolis is considered the second most fundamental equation only next to the Newton's second law in the analysis of navigation and guidance. Readers who are interested in the geometrical approach in the derivation of (7-14) may find it in other text books such as "Mechanics" by Symon (published by Addison Wesley). Some may find the geometrical approach simpler and easier to follow, while others may not. For this reason, an analytic approach is presented here to assist the comprehension in view of the importance of the theorem.

8.0 MATRIX FORM OF THE THEOREM OF CORIOLIS

The law of Coriolis in vector form (which we have derived previously) between Frames A and B follows:

$$P_A R = P_B R + W_{AB} \times R \,. \tag{8-1}$$

Coordinating (expressing components) in Frame B:

$$(P_A R)^B = (P_B R)^B + W_{AB}^B \times R^B \,. \tag{8-2}$$

Now, referring to (8-2):

$$
\begin{aligned}
(P_A R)^B &= C_A^B (P_A R)^A \\
&= C_A^B (PR^A) \text{ because } (P_A R)^A = (PR^A)^A = PR^A \\
&= C_A^B \left[P\left(C_B^A R^B\right)\right] \\
&= C_A^B \left[\left(PC_B^A\right)R^B + C_B^A\left(PR^B\right)\right] \\
&= C_A^B (PC_B^A) R^B + C_A^B C_B^A PR^B \\
&= C_A^B (PC_B^A) R^B + PR^B
\end{aligned} \tag{8-3}
$$

since $C_A^B C_B^A = C_B^B = I$

Returning to (8-2), recall the following from Section 1:

$$W_{AB}^B \times R^B \Leftrightarrow [W_{AB}^{BK}]R^B \tag{8-4}$$

where for $W_{AB}^B = \left(W_x W_y W_y\right)^T$

$$W_{AB}^{BK} = \begin{pmatrix} 0 & -W_z & W_y \\ W_z & 0 & -W_x \\ -W_y & W_x & 0 \end{pmatrix}$$

which is the matrix representation of the vector cross–product.

Substituting (8-3) and (8-4) into (8-2)

$$C_A^B(PC_B^A)R^B + PR^B = PR^B + [W_{AB}^{BK}]R^B \tag{8-5}$$

noting that $(P_BR)^B = (PR^B)^B = PR^B$.

It follows:

$$C_A^B(PC_B^A)R^B = [W_{AB}^{BK}]R^B. \tag{8-6}$$

Since (8-6) has to be valid for all R^B, we have

$$C_A^B P(C_B^A) = W_{AB}^{BK} \tag{8-7}$$

premultiplying both sides of (8-7) by

$$(C_A^B)^{-1} = (C_A^B)^T = C_B^A$$

because C_A^B , being a coordinate transformation matrix (between two orthogonal frames), is an orthogonal matrix, we have

$$C_B^A C_A^B P(C_B^A) = C_B^A W_{AB}^{BK} \tag{8-8}$$

since $C_B^A C_A^B = C_A^A = I$

$$PC_B^A = C_B^A W_{AB}^{BK} \tag{8-9}$$

Equation (8-9) is a matrix form of the Equation of Coriolis.

Since

$$PC_B^A = \frac{d}{dt} C_B^A \cong \frac{C_B^A(n+1) - C_B^A(n)}{\Delta t} \tag{8-10}$$

where $C_B^A(n)$ denotes the value of C_B^A at nT.

We have from (8-9):

$$C_B^A(n+1) = C_B^A(n) + C_B^A(n) W_{AB}^{BK}(n) \Delta t \tag{8-11}$$

Equation (8-10) suggests that if we can measure W_{AB}^{BK} by gyros fixed to the body (such as the rocket structure), then we can update the orientation of the body-fixed frame relative to a reference frame (such as the I-frame) by updating the C_B^A matrix. This idea is used in the so-called strap-down INS where the gyros and accelerometers are fixed to the body instead of on the inertial platform, which is isolated from the body motion by a servo system with gimbals. It may also be used in error analysis of the IMU platform drift caused by the gyro-drifts rates.

9.0 ROTATIONAL FORM OF NEWTON'S SECOND LAW

The rotational form of Newton's second law is the basic starting equation essential to describing the operation of gyro, gyro-accelerometer, and base–motion isolation systems. One such system is the stable platform of the IMU of a LRBM, which is controlled by a servo-mechanism using gyros and gimbals.

The equation also leads to the so-called Euler equation, describing the dynamic behavior of the rigid body motion, which is used in the SDOF simulation model of the rocket body in flight.

Consider a point mass m located at point k. By definition the moment of momentum or "angular momentum" H_{Ik} of m about the point I (origin of the I-frame) is

$$H_{IK} = R_{IK} \times mP_I R_{IK} \ . \qquad (9\text{-}1)$$

The moment of force or "torque" M_{Ik} of external force on m at k about the point I is by definition the following:

$$M_{Ik} = R_{Ik} \times F_k \qquad (9\text{-}2)$$

Differentiating (9-1) WRT time and noting that the cross-product of two identical vectors is zero:

$$\begin{aligned} P_I H_{Ik} &= P_I(R_{Ik} \times mP_I R_{Ik}) \\ &= P_I R_{Ik} \times mP_I R_{Ik} + R_{Ik} \times mP_I^2 R_{Ik} \\ &= R_{Ik} \times mP_I^2 R_{Ik} \\ &= R_{Ik} \times F_k \end{aligned}$$

It follows in view of (9-2):

$$P_I H_{Ik} = M_{Ik} \qquad (9\text{-}3)$$

Summing up (9-3) for all particles (all k's), and defining H_I and M_I by

$$H_I \doteq \Sigma_k (R_{Ik} \times m_k P_I R_{Ik}) \qquad (9\text{-}4)$$

$$M_I \doteq \Sigma_k (R_{Ik} \times F_k) \qquad (9\text{-}5)$$

and with the understanding that F_k refers to the external forces only, we get

$$P_I H_I = M_I \ . \qquad (9\text{-}6)$$

This is the rotational form of Newton's second law. The law states that the time rate of change (WRT the I-frame) of the angular momentum about an inertial point is equal to the torque about the same inertial point.

By denoting the location of the center of mass by C, R_{Ik} may be expressed by a simple vector addition:

$$R_{Ik} = R_{IC} + R_{Ck} \tag{9-7}$$

Substituting (9-7) into (9-4) and (9-5) and then the results into (9-6), after some algebra (see for instance Reference (9-1)), we get the rotational form of Newton's second law about the center of mass (instead of the inertial point):

$$P_I H_C = M_C \tag{9-8}$$

where

$$H_c = \Sigma_k \left(R_{Ck} \times m_k P_I R_{Ck}\right) \tag{9-9}$$

= the angular momentum of the system of particles about the center of mass

$$M_C = \Sigma_k R_{Ck} \times F_k \tag{9-10}$$

= the external torque about the center of mass.

Equation (9-8) states that the applied torque about the center of mass is equal to the time rate of change (WRT the I-frame) of the angular momentum about the center of mass.

The rotational form of Newton's second law is valid for any system of particles whose internal forces are central, whether or not the body is rigid.

10. PROTOTYPE LINEAR ACCELEROMETER*

Consider a box (instrument) with a spring and a test mass m. The box is rigidly attached to the rocket structure as shown in Figure 10-1. The rocket is moving along a straight line under a constant acceleration.

Under this steady-state condition, the spring extends by a constant displacement x from P to Q along the unit vector U, because of the inertial reaction force on m, which is the reaction to the applied force exerted on P.

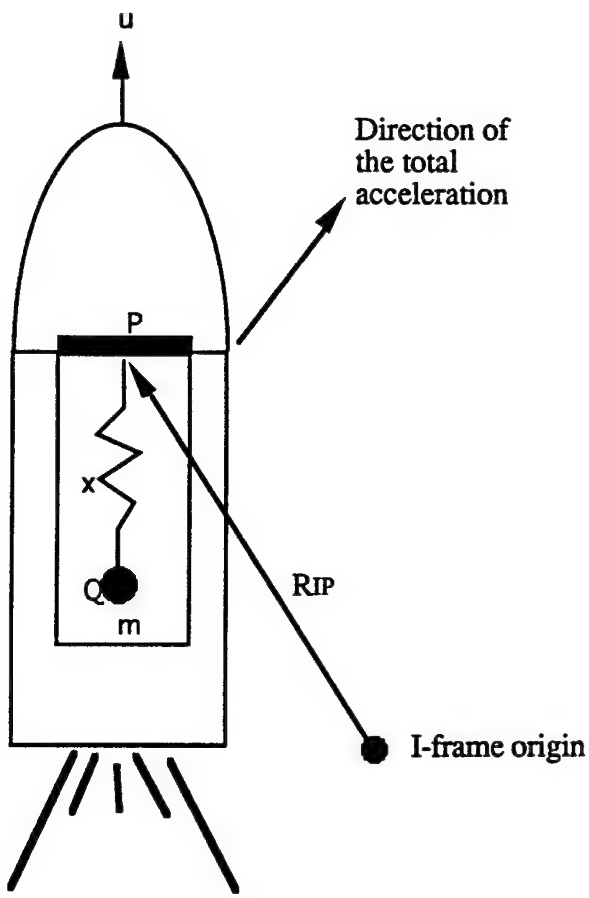

FIGURE 10-1. PROTOTYPE LINEAR ACCELEROMETER

The forces acting on m at point Q are the gravitational forces F_{GQ} (the gravitational force F_G at point Q), and the force -kx pulling the mass toward the point P by the restoring force of the spring.

Applying Newton's second law to m at Q with components expressed along the missile center line, by taking dot product with u:

$$mP_I^2R_{IQ} \cdot u = \left(F_{GQ} - k\underline{x}\right) \cdot u. \tag{10-1}$$

Since by vector additions:

$$R_{IQ} = R_{IP} + R_{PQ} \tag{10-2}$$

we get by substituting (10-2) into (10-1):

$$\left(mP_I^2R_{IP} + mP_I^2R_{PQ}\right) \cdot u = \left(F_{GQ} - k\underline{x}\right) \cdot u \tag{10-3}$$

Under steady-state conditions R_{PQ} is constant, which makes P_IR_{PQ} and, therefore, $P_I^2R_{PQ}$ equal to zero. It follows from (10-3):

$$P_I^2R_{IP} \cdot u = \left(\frac{F_{GQ}}{m} - \frac{k}{m}\underline{x}\right) \cdot u \tag{10-4}$$

Noting that $\frac{F_{GQ}}{m}$ is the gravitational acceleration g at Q (denoted by g_Q), we have from (10–4):

$$P_I^2R_{IP} \cdot u = \left(g_Q - \frac{k}{m}x\right) \cdot u. \tag{10-5}$$

Applying Newton's second law to the whole box considered as a rigid body (instrument with the pivot point at P), noting that the acceleration at P has to be caused by either gravitational specific force g_P at P or nongravitational specific force f_{NGP} at P, and taking the component along the u direction:

$$\left(P_I^2R_{IP}\right) \cdot u = \left(g_P + f_{NGP}\right) \cdot u \tag{10-6}$$

Equating (10-5) to (10-6) and rearranging

$$f_{NGP} \cdot u = -\frac{k}{m}x + \left(g_Q - g_P\right) \cdot u \tag{10-7}$$

Now both g_P and g_Q are gravitational accelerations (gravitational force per unit mass) several thousand miles away from the center of the earth at points P and Q that are less than only a few centimeters apart. Thus, g_Q and g_P are essentially equal and $g_Q - g_P = 0$.

It follows from (10-7)

$$f_{NGP} \cdot u = -\frac{k}{m} x \tag{10-8}$$

Equation (10-8) shows that the mass-spring box (an idealized accelerometer) measures the externally applied nongravitational force per unit mass (called nonfield "specific force"), which is equal to the force (per unit mass) that the test mass m exerts on its support at P through the spring. That is, the spring displacement x is the measure of nongravitational force per unit mass (specific force) along the direction u at point P. Since x is preportional to $-f_{NGp} \cdot u$, we should say, more exactly, that the accelerometer measures the negative of the nonfield specific force along (u).

It is important to note that the accelerometer (a misnomer in a strict sense) measures nongravitational specific force f_{NG}, not $f_{NG} + f_G$, Newton's second law states that

$$P_I^2 R_{IP} = \sum f = f_G + f_{NG} .$$

Thus, in the implementation of the navigation system that relies on the onboard accelerometers, we have to always add the gravitational acceleration (computed generally based on the inverse-square gravity model) to obtain the total acceleration as shown below in Figure 10-2.

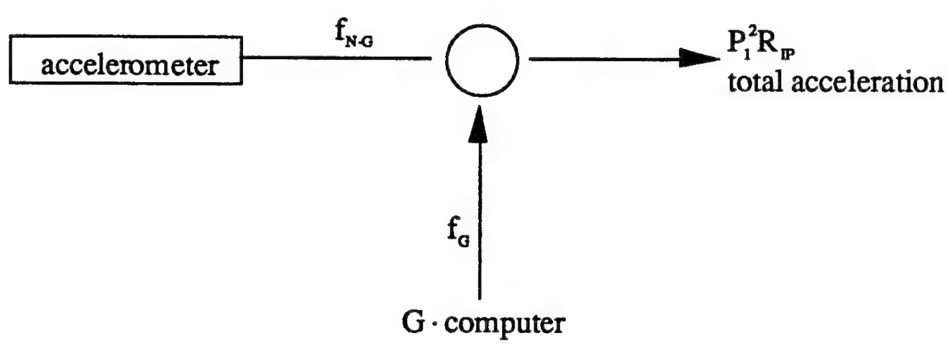

FIGURE 10-2. COMPUTATION OF TOTAL ACCELERATION

For some, the fact that the accelerometer onboard a rocket in motion under a gravitational field measures only the nongravitational specific force may seem counter-intuitive.

Referring to the first figure of this section, both points P and Q located more than 6,000 km away from the center of the earth are only less than a few centimeters apart. This means that the gravitational force per unit mass at these two points is essentially equal.

Thus, we can see that the effect of gravitational force on the displacement of the spring is essentially zero. In another words, the accelerometer is incapable of sensing the gravitational force and therefore cannot measure it. Now, let us consider the combined effect of thrust and aerodynamic forces along the direction u. This will push the rocket structure forward along with the pivot-point P, which is fixed to the structure. By Newton's first law, the mass M located at Q will resist this change and try to remain behind, thus stretching the spring by inertial reaction. Now, we can see why the accelerometer senses and measures the effect of nongravitational force only excluding that of gravitational force.

11.0 PULSE INTEGRATED PENDULOUS ACCELEROMETER (PIPA)*

Consider a physical pendulum of mass m with the moment of inertia J and with the center of mass (CM) located at distance L from the pivot point P. It is damped with the damping coefficient C.

The vehicle with the pendulum (pivot point P) fixed to it is under acceleration by the combined effect of both gravitational and nongravitational forces. The nongravitational portion of the acceleration is acting in the direction labeled as IA in Figure 11-1.

As explained in the case of linear accelerometer, the gravitational force per unit mass $\left(\frac{F_G}{m} = f_G\right)$ or gravitational acceleration at pivot point is essentially identical to that at the CM of the pendulum.

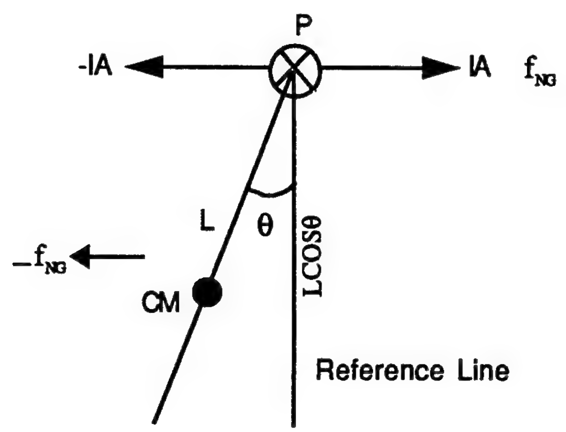

FIGURE 11-1. PHYSICAL PENDULUM

Thus, the gravitational acceleration does not contribute to the swinging of the pendulum. However, the nongravitational specific force f_{ng} (thrust and aerodynamic force) is applied to the point p, which is fixed to the vehicle, but not to CM (unlike the gravitational force that is exerted on both p and CM). As p accelerates along IA, m, according to Newton's first law, tries to remain stationary in the inertial space, thus making angle θ with the reference line.

Applying the rotational form of Newton's second law about a pivot axis (into the paper):

$$J\ddot{\theta} = \Sigma M \tag{11-1}$$

*This section is adapted from Chapter 1.0 (the introduction) of MIT Instrumentation Laboratory Report R-574 (on PIPA) by Kee Soon Chun, March, 1967.

The applied torques ΣM consists of the following three parts:

- Torque caused by the inertial reaction force $-F_{NG} = -m\frac{F_{NG}}{m} = -mf_{NG}$ about IA axis. The negative sign is necessary because the inertial reaction force is in the opposite direction from the applied F_{NG}. This torque is from the diagram, $L\cos\theta(-mf_{NG}) = -mL\cos\theta f_{NG}$

- Damping (dynamic friction) torque $-C\dot{\theta}$ (with negative sign because it opposes the motion).

- Rebalancing torque generated by the torque generator M_{tg}, which drives the angular deviation θ to zero.

In actual operation, the angular deviation θ of the pendulum from the reference line is limited to very small angles and constantly rebalanced by the rebalancing torque generated by the torque generator located at the pivot point (not shown in the diagram).

Thus, from (11-1), summing up torques:

$$J\ddot{\theta} = M_{tg} - mL\cos\theta f_{NG} - C\dot{\theta} \tag{11-2}$$

since θ is limited to very small angles $\cos\theta \cong 1$. Thus

$$J\ddot{\theta} + C\dot{\theta} = M_{tg} - mLf_{NG} \tag{11-3}$$

Integrating with respect to time for the duration of Δt:

$$J\int_0^{\Delta t}\ddot{\theta}dt + C\int_0^{\Delta t}\dot{\theta}dt = \int_0^{\Delta t}M_{tg}dt - mL\int_0^{\Delta t}f_{NG}dt \tag{11-4}$$

Because of the nature of the rebalancing operation, the angular displacement constantly reverses its direction. This causes the average value of $\dot{\theta}$ (and average value of $\ddot{\theta}$ as well) to approach zero for the time duration much greater than the period T_{tg} of each rebalancing torque pulse, which is indeed the case in actual application.

Thus, for $\Delta t >> T_{tg}$

$$\int_0^{\Delta t}\dot{\theta}dt = \int_0^{\Delta t}\ddot{\theta}dt = 0 \tag{11-5}$$

It follows from (11-4) with (11-5):

$$\int_0^{\Delta t} M_{tg} dt = mL \int_0^{\Delta t} f_{NG} dt . \tag{11-6}$$

In the actual system, M_{tg} is proportional to the constant DC current I, through the torque generator (with the permanent magnet). Thus,

$$M_{tg} = S_{tg} I \tag{11-7}$$

where S_{tg} is the torque generator sensitivity. It follows,

$$\int_0^{\Delta t} M_{tg} dt = \int_0^{\Delta t} S_{tg} I dt = S_{tg} I \Delta t . \tag{11-8}$$

By making the integration duration Δt sufficiently small (while keeping $\Delta t >> T_{tg}$) relative to the thrust and aero–dynamic force variation, we may treat f_{ng} constant during Δt. Thus,

$$\int_0^{\Delta t} f_{NG} dt = f_{NG} \Delta t = \Delta V_{NG} \tag{11-9}$$

where ΔV_{ng} is the velocity increment caused by f_{ng} during Δt. Substituting (11-8) and (11-9) into (11-6):

$$S_{tg} I \Delta t = mL \Delta V_{NG}$$

or

$$\Delta V_{NG} = \frac{S_{tg}}{mL} I \Delta t . \tag{11-0}$$

Thus, ΔV_{ng} is proportional to I Δt (integration of current), or the amount of electric charge passed through the DC torque motor during Δt. If we use electric pulses

with constant magnitude (instead of DC current), the summation (integration) of pulses is also equal to the amount of electric charges. Because ΔV_{NG} is the integration of f_{NG} (which, in turn, proportional to the summation (integration) of current pulses) during Δt, this type of instrument is called pulse integrated pendulous accelerometer (PIPA).

From (11-10) with Δt fixed

$$d(\Delta V_{NG}) = \frac{S_{tg}\Delta t}{mL} dI$$

$$= \frac{\Delta V_{NG}}{I} dI \quad \text{(using (10) again)} \tag{11-11}$$

or

$$\frac{d(\Delta V_{NG})}{\Delta V_{NG}} = \frac{dI}{I} \tag{11-12}$$

Equation (11-12) shows that one part per million error in the DC current measurement would cause the corresponding one part per million error in the estimation of the velocity increment ΔV_{ng}. In actual implementation, the constant current source for the DC motor is achieved, in essence, by applying a constant voltage from a zener diode across a precision resister. But, both zener diodes and precision resistors are not only temperature sensitive but also deteriorate with time. For this reason, to achieve more accuracy, PIPA is replaced by PIGA in the newer long-range ballistic missile. PIGA is discussed in a later section.

12.0 ROTATIONAL MOTION OF THE MISSILE BODY

In the previous section on the rotational term of Newton's second law (Section 9.0), the following equation has been derived:

$$P_I H_C = M_C \tag{12-1}$$

where H_c is the angular momentum about the center of mass, M_c is the applied moment (torque) about the center of mass, and $P_I(\cdot)$ is $\frac{d}{dt}(\cdot)$ in the I-frame.

Applying the theorem of Coriolis to (12-1) between the I-frame and a body fixed body (B)-frame of the missile and expressing the components in the B-frame:

$$(P_I H_C)^B = P_B H_C^B + W_{IB}^B \times H_C^B = M_C^B \tag{12-2}$$

As shown in Section 9.0, the angular momentum H_c is given by

$$H_C = \sum_K m_k R_{CK} \times P_I R_{CK} \tag{12-3}$$

Applying the theorem of Coriolis to R_{ck} between the I-frame and the B-frame, which rotates relative to the I-frame with the angular velocity W_{IB}, we have:

$$\begin{aligned} P_I R_{CK} &= P_B R_{CK} + W_{IB} \times R_{CK} \\ &= W_{IB} \times R_{CK} \end{aligned}$$

since $P_B R_{CK} = 0$ because R_{CK} is fixed in the B-frame. (Strictly speaking, rocket exhaust gases cause the center of mass of the rocket to shift, and R_{CK} is not fixed in the B-frame. The derivation here assumes that the shift of the center of mass is negligible for the time span of this discussion.)

Substituting $W_{IB} \times R_{CK}$ for $P_I R_{CK}$ in (12-3), we have

$$H_c = \sum_K m_k R_{CK} \times (W_{IB} \times R_{CK}) \tag{12-3b}$$

Using a vector identity:

$$\begin{aligned} V_1 \times (V_2 \times V_3) &= (V_1 \cdot V_3) V_2 - (V_1 \cdot V_2) V_3 \\ &= (V_1 \cdot V_1) V_2 - (V_1 \cdot V_2) V_1 \quad \text{for } V_1 = V_3 \end{aligned}$$

we get from (12-3b):

$$H_C = \sum m_k (R_{CK} \cdot R_{CK}) W_{IB} - \sum m_k (R_{CK} \cdot W_{IB}) R_{CK} \,. \tag{12-4}$$

At this point, we define the x, y, z axis of the B-frame with the origin at center of gravity (CG) of the missile as shown in Figure 12-1.

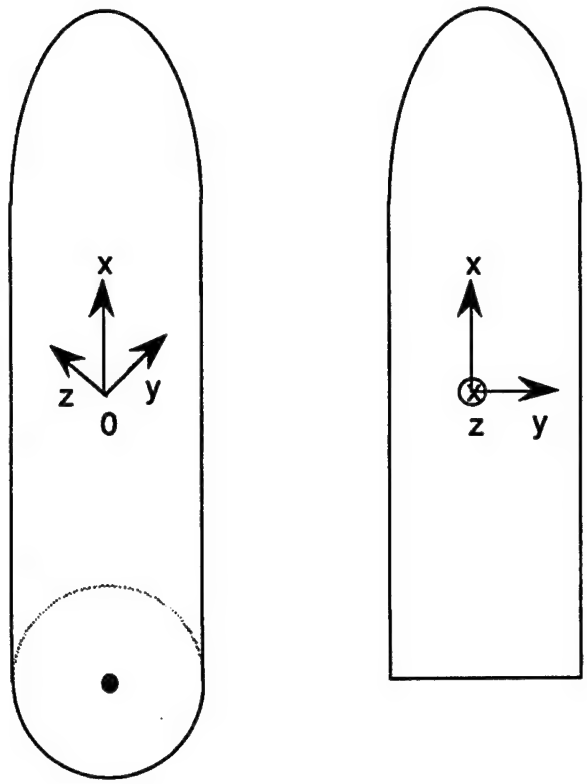

FIGURE 12-1. B-FRAME FIXED TO THE MISSILE

where the axes are defined as :

X axis— Along the missile center line

Y axis —Perpendicular to the center line at the CG

Z axis —Perpendicular to both the x and y axis at the origin so as to form an orthogonal triad by the right-hand rule ($X \otimes Y = Z$)

Now consider the transversal cross-section of the missile on the y-z plane as shown below in Figure 12-2.

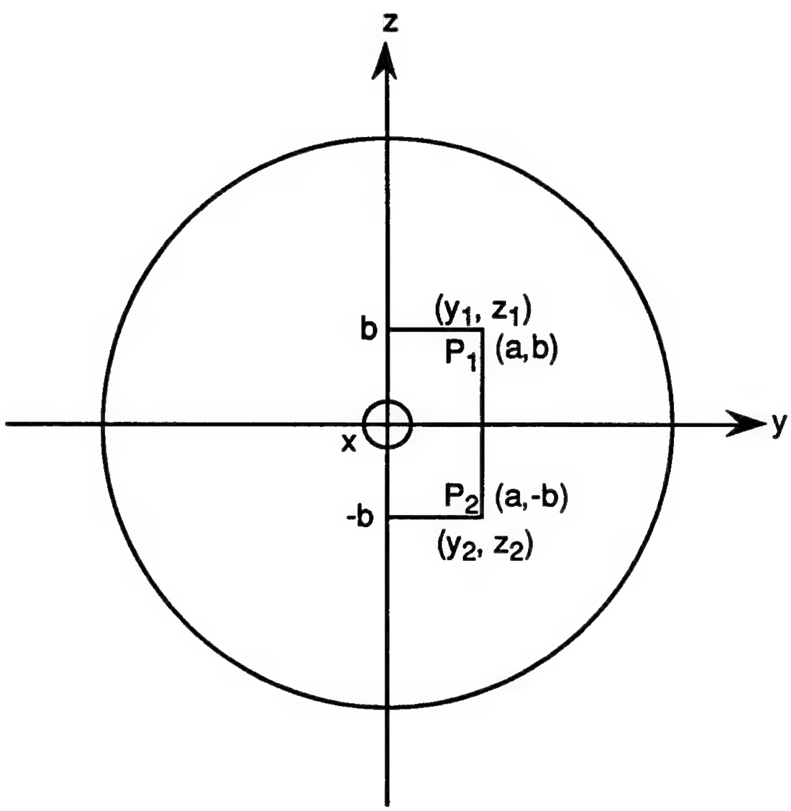

FIGURE 12-2. CROSS-SECTION (y-z PLANE) OF THE MISSILE

We see that for any point $P_1(y_1,z_1)$, there exist $P_2(y_2,z_2)$,within the missile such that

$$y_1 z_1 + y_2 z_2 = ab + (-ab) = 0 .$$

Extending this situation to cover the entire volume of the missile,

$$\sum_K y_K z_K = 0 \tag{12-5}$$

Next, consider a longitudinal cross-section on the x-y plane as shown below in Figure 12-3.

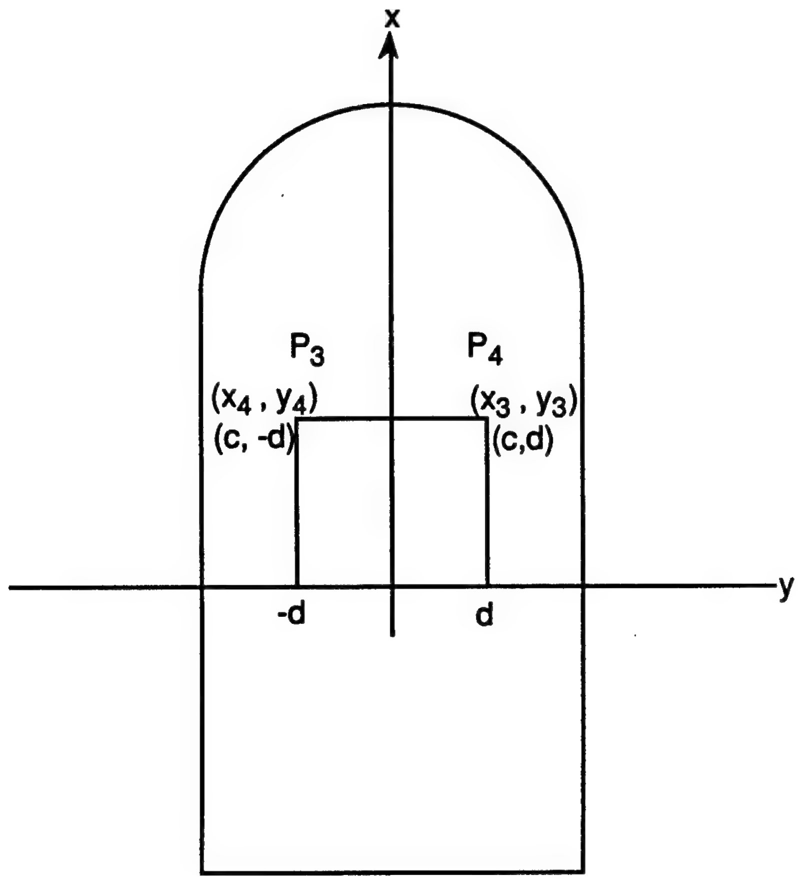

FIGURE 12-3. CROSS-SECTION (x - y PLANE) OF THE MISSILE

We see that for any point $P_3(x_3, y_3)$, there exist $P_4(x_4, y_4)$, within the missile such that $x_3y_3 + x_4y_4 = cd + (-cd) = 0$.

Therefore, for the entire volume of the missile,

$$\sum_K x_K y_K = 0 . \tag{12-6}$$

Similarly on the xz plane:

$$\sum_K x_K z_K = 0 . \tag{12-7}$$

By denoting R^B_{CK} and W^B_{IB} in (12-4) by

$$R_{CK}^{B} = \begin{pmatrix} x_K \\ y_K \\ z_K \end{pmatrix} \text{ and } W_{IB}^{B} = \begin{pmatrix} W_x \\ W_y \\ W_z \end{pmatrix} \tag{12-8}$$

we have

$$\sum_k m_k \left(R_{CK}^{B} \cdot R_{CK}^{B}\right) W_{IB}^{B} = \sum m_k \left(x_K^2 + y_K^2 + z_K^2\right) \begin{pmatrix} W_x \\ W_y \\ W_z \end{pmatrix} \tag{12-9}$$

Using (12-5), (12-6), and (12-7) in the second term of (12-4):

$$\sum_K m_k \left(R_{CK}^{B} \cdot W_{IB}^{B}\right) R_{CK}^{B} = \sum_K m_k (x_K W_X + y_K W_Y + z_K W_Z) \begin{pmatrix} x_K \\ y_K \\ z_K \end{pmatrix}$$

$$= \begin{pmatrix} \sum m_k x_K^2 W_x \\ \sum m_k y_K^2 W_y \\ \sum m_k z_K^2 W_z \end{pmatrix}. \tag{12-10}$$

Substituting (12-9) and (12-10) into (12-4):

$$H_C^B = \begin{pmatrix} \sum m_k \left(y_K^2 + z_K^2\right) W_x \\ \sum m_k \left(x_K^2 + z_K^2\right) W_y \\ \sum m_k \left(x_K^2 + y_K^2\right) W_z \end{pmatrix} = \begin{pmatrix} J_x W_x \\ J_y W_y \\ J_z W_z \end{pmatrix} \tag{12-11}$$

which may be written as :

$$H_C^B = \begin{pmatrix} J_x & O & O \\ O & J_y & O \\ O & O & J_z \end{pmatrix} \begin{pmatrix} W_x \\ W_y \\ W_z \end{pmatrix} = JW \tag{12-12}$$

where

$$\begin{aligned} J_x &= \sum m_K \left(y_K^2 + z_K^2\right) \\ J_y &= \sum m_K \left(x_K^2 + z_K^2\right) \\ J_z &= \sum m_K \left(x_K^2 + y_K^2\right) \end{aligned} \quad \text{and } J = \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{pmatrix}. \tag{12-13}$$

In (12-12) and (12-13), we note that all off-diagonal elements of the J-matrix, which is called the moment of inertia tensor, are all zeros, that is, the J-matrix is a diagonal matrix, thus greatly simplifying analysis and computations. The choice of body axes, which result in the diagonal J-matrix is called the "principal axes." What seems to be a natural choice of axes in our case happens to coincide with the principle axes.

For analytical methods (which are not simple) for determining the principal axes, refer to Mechanics by K.R. Symon or Classical Mechanics by H. Goldstein.

Two proven theorems for selecting the principal axes follow:

- Any axis of symmetry of a body is a principal axis such as the x-axis in our case.
- Any plane of symmetry of a body is perpendicular to a principal axis. In our case, the plane (containing the x-axis), which is perpendicular to the y-axis is perpendicular to the principal axis, thus making the y-axis the principal axis. Similarly for the z-axis.

Now referring to (12-2), we have, using (12-11):

$$P_B H_C^B = \frac{d}{dt}\begin{pmatrix} J_x W_x \\ J_y W_y \\ J_z W_z \end{pmatrix} = \begin{pmatrix} J_x \dot{W}_x \\ J_y \dot{W}_y \\ J_z \dot{W}_z \end{pmatrix} \tag{12-14}$$

since J_x, J_y, J_z are constants in view of (12-13) for constant fixed mass missiles. (J_x, J_y, J_z are not constants for variable mass missiles such as real-world missiles in flight.)

And,

$$W_{IB}^B \times H_C^B = \begin{vmatrix} i & j & k \\ W_x & W_y & W_z \\ J_x W_x & J_y W_y & J_z W_z \end{vmatrix}$$

$$= \begin{pmatrix} J_z W_y W_z - J_y W_y W_z \\ J_x W_x W_z - J_z W_x W_z \\ J_y W_x W_y - J_x W_x W_y \end{pmatrix} = \begin{pmatrix} (J_z - J_y) W_y W_z \\ (J_x - J_z) W_x W_z \\ (J_y - J_x) W_x W_y \end{pmatrix}. \tag{12-15}$$

Substituting (12-14) and (12-15) into (12-2), and expressing the components of M_C^B by M_x, M_y, M_z we have

$$J_x\dot{W}_x + (J_z - J_y)W_yW_z = M_x \tag{12-16a}$$

$$J_y\dot{W}_y + (J_x - J_z)W_xW_z = M_y \tag{12-16b}$$

$$J_z\dot{W}_y + (J_y - J_x)W_xW_y = M_z\,. \tag{12-16c}$$

Equation (12-16) is called Euler's equations (of motion) for a rigid body (along the principal axes).

Equation (12-16) may be solved for $W_x(t)$, $W_y(t)$, $W_z(t)$, (which are the angular velocities of the missile body relative to the inertial space) in terms of M_x, M_y, M_z. M_x, M_y, and M_z are the externally applied torques generated by the reaction force to thrust (which is, by custom, simply called thrust) and the aerodynamic force about the CG of the missile (known as a function of time in simulation).

In Section 17.0 on the matrix form of the theorem of Coriolis, we have derived

$$\frac{d}{dt}C_B^I = C_B^I W_{IB}^{BK} \tag{12-17}$$

(with the superscript A of (12-9) in that section replaced by I) where W_{IB}^{BK} is a 3X3 matrix given by

$$W_{IB}^{BK} = \begin{pmatrix} O & -W_z & W_y \\ W_z & O & -W_x \\ -W_y & W_x & O \end{pmatrix}. \tag{12-18}$$

With $W_x(t)$, $W_y(t)$, $W_z(t)$ from (12-18), then Equation (12-17) may be solved for $C_B^I(t)$

$$C_B^I(t) = \begin{pmatrix} C_{11}(t) & C_{12}(t) & C_{13}(t) \\ C_{21}(t) & C_{22}(t) & C_{23}(t) \\ C_{31}(t) & C_{32}(t) & C_{33}(t) \end{pmatrix} \tag{12-19}$$

which is the coordinate transformation matrix from the B-frame to the I-frame, thus specifying the angular orientation of the missile in the inertial space.

Newton's second law applied to the CG of the missile and expressed component-wise in the inertial frame is:

$$P_I^2\ddot{X}_I = f_{GX} + f_{NGX} \tag{12-20a}$$

$$P_I^2\ddot{Y}_I = f_{GY} + f_{NGY} \tag{12-20b}$$

$$P_I^2\ddot{Z}_I = f_{GZ} + f_{NGZ} \tag{12-20c}$$

where $P_I^2(\cdot)=\dfrac{d^2}{dt^2}(\cdot)$ in the inertial space, f_g is the gravitational force per unit mass, and f_{NG} is the nongravitational specific (per unit mass) force which includes both thrust and aerodynamic forces. The f_{NG} is most conveniently expressed in the B-frame and transformed to the inertial frame.

The three equations of (12-20) completely specify the translational motion (position and velocity) of the missile's CG along the inertial X_I, Y_I, Z_I axes, while another three equations (12–16) completely specify the angular orientation (which is the integration of the angular rates) of the missile in the inertial space. For this reason, the mathematical model for simulation based on these six equations are called a six degree-of-freedom (6 DOF) model.

If we are concerned only with the translational motion of the CG, ignoring the missile orientation, only the three equations of 20's are used, and its model is called a 3 DOF model.

Since the motion of CG of the missile is the same as the motion of the point mass in which the entire mass of the missile is concentrated at the CG, it is also called the point mass model.

13.0 THE GYROS*

The rotational form of Newton's second law about the center of mass *C* is, as derived previously :

$$P_I H_C = M_C . \tag{13-1}$$

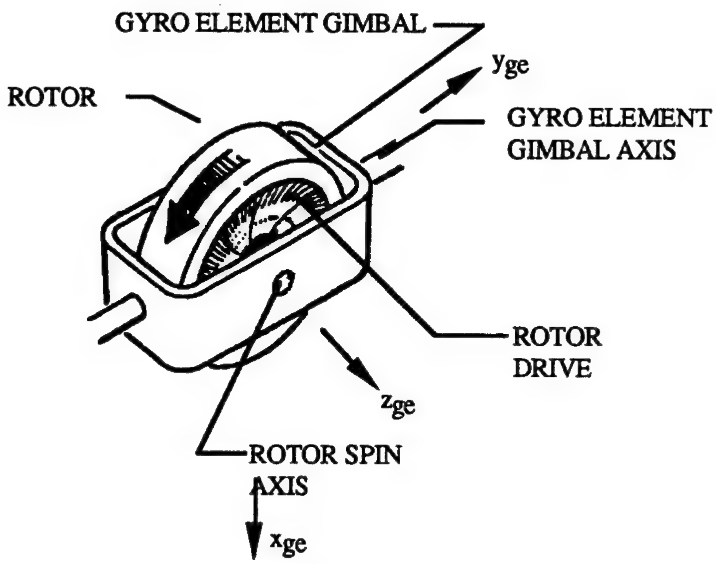

FIGURE 13-1. GYRO ELEMENT

The assembly (the spinning rotor and its support gimbal) shown in Figure 13-1 is called "gyro element" (*ge*), with axis of the *ge* frame as shown in the figure. Replacing subscript *C* by *ge* (*cm* of *ge*) in (13-1) and, applying the law of Coriolis between I-frame and *ge* frame,

$$P_I H_{ge} = P_{ge} H_{ge} + W_{i\text{-}ge} \times H_{ge} = M_{ge} \tag{13-2}$$

where the subscript i-ge is the same as I-ge.

The angular momentum of the gyro element H_{ge} consists of the spin part H_s (constant) and nonspin part H_{ns}, that is :

$$H_{ge} = H_s + H_{ns} \tag{13-3}$$

where H_{ns} is the angular momentum vector of the gyro element other than that of the spin rotor.

Substituting (13-3) into (13-2):

$$P_{ge}H_s + P_{ge}H_{ns} + W_{i\cdot ge} \times (H_s + H_{ns}) = M_{ge} \,. \tag{13-4}$$

In (13-4), $P_{ge}H_s = 0$ because H_s is constant and fixed in *ge* frame, and therefore

$$W_{i\cdot ge} \times (H_s + H_{ns}) \cong W_{ie} \times H_s$$

because

$$H_s >> H_{ns} \,.$$

It follows that

$$P_{ge}H_{ns} + W_{i\cdot ge} \times H_s = M_{ge} \,. \tag{13-5}$$

When $P_{ge}H_{ns}$ is negligible compared to $W_{i\cdot ge} \times H_s$, (13-5) reduces to

$$W_{i\cdot ge} \times H_s = M_{ge} \tag{13-6}$$

Equation (13-6) is the approximate performance equation for the practical gyro. The geometric interpretation of (13-6) is depicted in Figure 13-2 with axes defined in Figure 13-3. That is, the spin angular momentum H_s precesses (rotates) relative to the inertial space with the angular velocity of *ge* (relative to I-frame), $W_{i\cdot ge}$, in an attempt to align itself with the applied torque vector M_{ge} about the center of mass of *ge*.

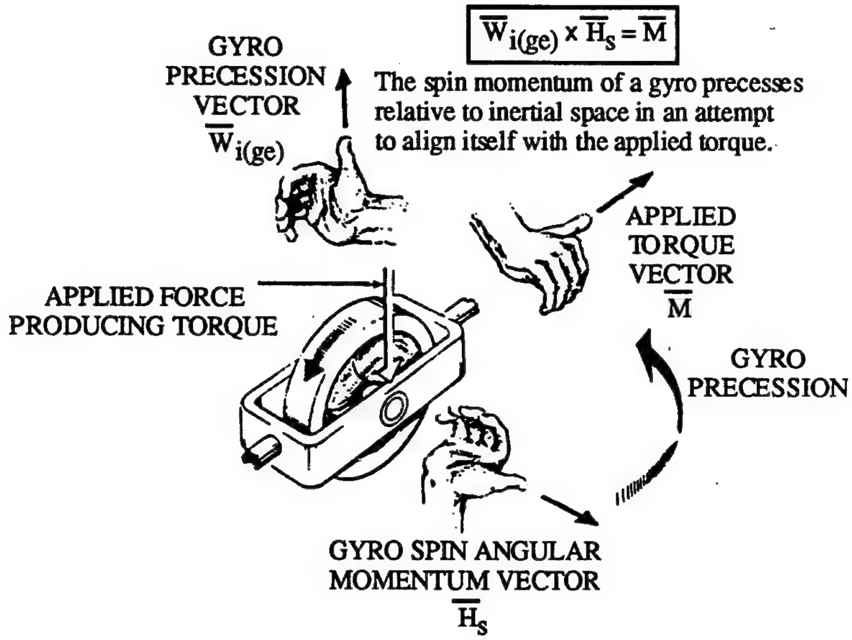

FIGURE 13-2. REPRESENTATION OF THE BASIC LAW OF MOTION

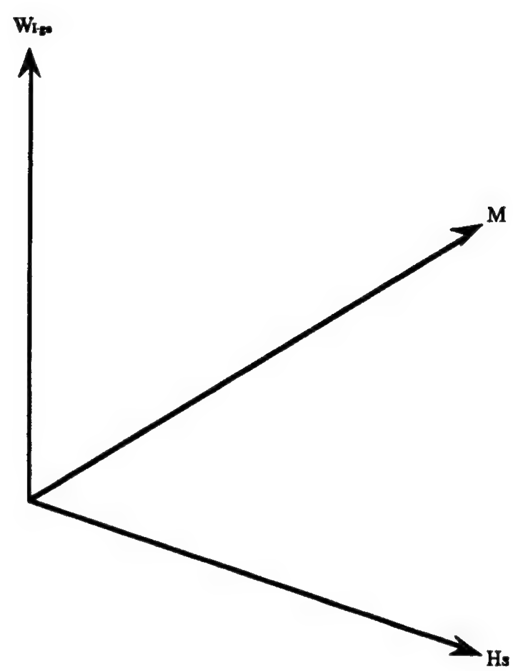

FIGURE 13-3. DEFINITION OF THE PRACTICAL GYROSCOPE AXES

Referring to (13-5), we express each vector by its components by

$$W_{i-ge}^{ge} = \begin{pmatrix} W_x \\ W_y \\ W_z \end{pmatrix}^{ge} \quad ; \quad H_s^{ge} = \begin{pmatrix} 0 \\ 0 \\ H_s \end{pmatrix}$$

and

$$P_{ge} H_{ns} = P_{ge} \begin{pmatrix} J_x \; W_x \\ J_y \; W_y \\ J_z \; W_z \end{pmatrix}^{ge} = \begin{pmatrix} J_x \; P_{ge} \; W_x \\ J_y \; P_{ge} \; W_y \\ J_z \; P_{ge} \; W_z \end{pmatrix}^{ge} = \begin{pmatrix} J_x \; P \; W_x \\ J_y \; P \; W_y \\ J_z \; P \; W_z \end{pmatrix}$$

$$M_{ge} = \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix}^{ge}$$

where J_x, J_y, and J_z are the moments of inertia of gyro-element about its x, y, and z axes. (J is used instead of I, which is used for identity matrix.) Substituting these into (13-5), and performing the cross-product $W_{i \cdot ge} \times H_s$, we get the following:

$$J_x p W_x + H_s W_y = M_x \tag{13-7a}$$

$$J_y p W_y - H_s W_x = M_y \tag{13-7b}$$

$$J_z p W_z + 0 = M_z \,. \tag{13-7c}$$

Understanding the components are expressed in ge–frame.

Of these, only 13-7b is of interest to us, which is repeated in a modified form with original subscripts for W_x and W_y.

$$P(J_y W_{(i \cdot ge)y}) = M_y + H_s W_{(i \cdot ge)x} \,. \tag{13-8}$$

Equation 13-8 shows that the time rate of change of the angular momentum about the y-axis of gyro-element is equal to the sum of the applied torque about the y-axis and the gyro-scopic torque created by the interaction of the gyro's spin angular momentum H_s and the angular velocity of the gyro-element about its x-axis.

Some authors refer to this gyroscope torque as the "reaction" torque caused by the $W_{(I \cdot ge)x}$. However, it is not "reaction" in the sense of Newton's third law because $H_s W_{(I \cdot ge)x}$ acts in the y-direction, while $W_{(I \cdot ge)x}$ is in the x-direction, which is perpendicular to the y-direction.

14.0 SDOF GYRO

In an SDOF gyro, the gyro-element (ge) consisting of the rotor and its gimbal is encased inside a case called the gyro-unit (gu), in such a way that the *ge* gimbal axis is co-axial with *gu* case's longitudinal (cylindrical) axis, which is called the output axis (OA), as shown in Figure 14–1. Thus, this type of gyro has no gimbal other than the ge-gimbal, which holds the spin-rotor. So, in the SDOF gyro, the gyro spin vector (called the *H* vector) denoting the angular momentum of the spin-rotor, is allowed to precess (rotate) about the output axis of the *gu* (case) only.

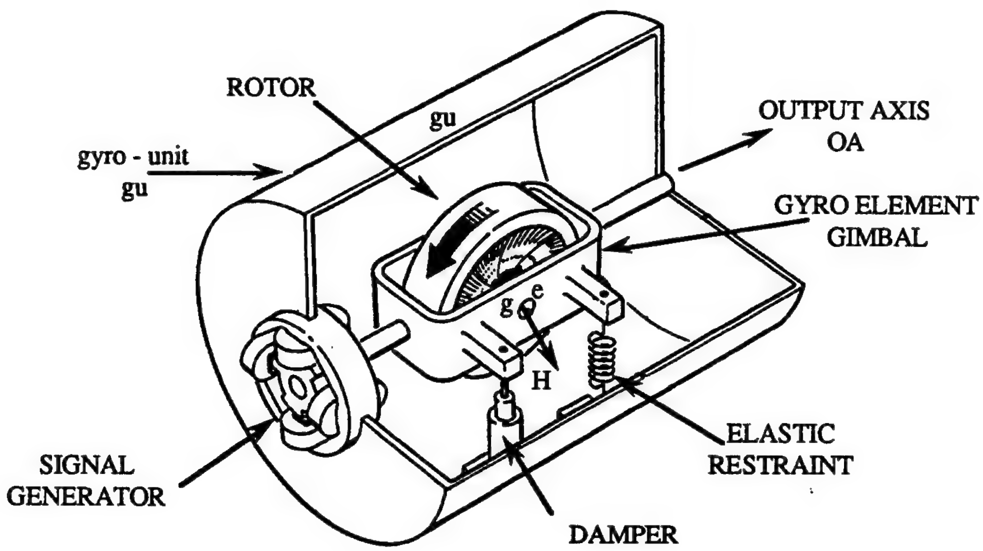

FIGURE 14-1. ESSENTIAL ELEMENTS OF A SDOF GYROSCOPE

The *ge* frame and *gu* frame are defined and depicted in Figures 14-2 and 14-3. Both frames share the same origin and the same *Y* axis, (i.e., $Y_{ge} = Y_{gu}$). The *ge* frame with spin axis along Z_{ge} rotates about the common axis of $Y_{ge} = Y_{gu}$ making angle α from Z_{gu}(SRA) to Z_{ge} (SA).

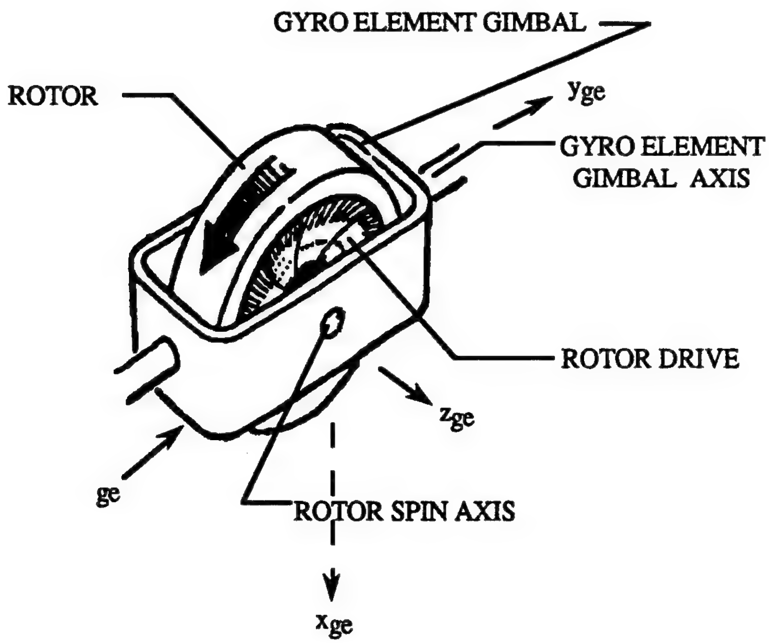

FIGURE 14-2. THE GYROELEMENT

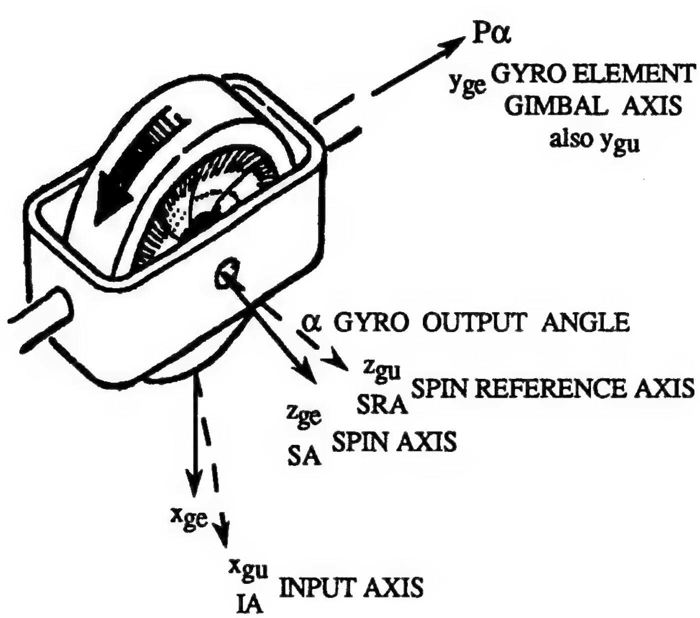

FIGURE 14-3. GYRO ELEMENT COORDINATE AXES

X_{gu} along IA (input axis) of gyro-unit (case)
Y_{gu} along OA (output/precession) axis of *gu* (case)

Z_{gu} along spin reference axis (SRA) of *gu* (case)

When $\alpha = 0$, $Z_{gu} = Z_{ge}$ or SRA = SA.

α is the angle the spin axis (SA) makes WRS the SRA, which coincides with the Z_{gu}) as a result of the rotation of the gyro-element about OA.

Now

$$W_{I\cdot ge} = W_{I\cdot gu} + W_{gu\cdot ge} . \tag{14-1}$$

Expressing (14-1) in the *ge* frame:

$$W^{ge}_{I\cdot ge} = C^{ge}_{gu} W^{gu}_{I\cdot gu} + W^{ge}_{gu\cdot ge} . \tag{14-2}$$

Referring to Figure 14-3 and noting that for a small angle α, $\sin\alpha \cong \alpha$ and $\cos\alpha \cong 1$,

$$X_{ge} = X_{gu}\cos\alpha + Z_{gu}\sin\alpha = X_{gu} + \alpha Z_{gu} \tag{14-3a}$$

$$Y_{ge} = Y_{gu} \tag{14-3b}$$

$$Z_{ge} = -X_{gu}\sin\alpha + Z_{gu}\cos\alpha \cong -X_{gu}\alpha + Z_{gu} . \tag{14-3c}$$

From the above equations,

$$\begin{pmatrix} X_{ge} \\ Y_{ge} \\ Z_{ge} \end{pmatrix}^{ge} = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ -\alpha & 0 & 1 \end{pmatrix}^{ge}_{gu} \begin{pmatrix} X_{gu} \\ Y_{gu} \\ Z_{gu} \end{pmatrix}^{gu} . \tag{14-4}$$

Thus, C^{ge}_{gu} in (14-2) is given by

$$C^{ge}_{gu} = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ -\alpha & 0 & 1 \end{pmatrix} . \tag{14-5}$$

We designate the components of $W^{ge}_{I\cdot ge}$ and $W^{gu}_{I\cdot gu}$ as follows:

$$W^{ge}_{I-ge} = \begin{pmatrix} W_x \\ W_y \\ W_z \end{pmatrix} \tag{14-6}$$

$$W^{gu}_{I \cdot gu} = \begin{pmatrix} W_{IA} \\ W_{OA} \\ W_{SRA} \end{pmatrix}. \tag{14-7}$$

Now, *ge* can rotate with respect to *gu* about $Y_{ge} = Y_{gu} = OA$ only, creating the angle α.

$$W^{ge}_{gu \cdot ge} = \begin{pmatrix} 0 \\ P\alpha \\ 0 \end{pmatrix} \tag{14-8}$$

Substituting (14-5), (14-6), (14-7), and (14-8) into (14-2) :

$$W_x = W_{IA} + \alpha W_{SRA} \tag{14-9}$$

$$W_y = W_{OA} + P\alpha \tag{14-10}$$

$$W_z = -\alpha W_{IA} + W_{SRA}. \tag{14-11}$$

The applied torque M_y about the OA ($Y_{gu} = Y_{ge}$ axis), consists of :

$-K\alpha$—Elastic torque proportional to angular displacement α and opposing α

$-CP\alpha$—Damping (dynamic–friction) torque proportional to angular velocity $P\alpha$, and opposing $P\alpha$

M_{tg}—Commanded torque applied by the torque generator located on the output axis

Thus, we may write

$$M_y = -K\alpha - CP\alpha + M_{tg} \tag{14-12}$$

By designer's choice, when $K\alpha$ predominates (relative to $CP\alpha$ term), the instrument becomes a rate gyro. If the $CP\alpha$ term predominates, the instrument becomes a (rate) integrating gyro. This is explained below:

In Section 12.0, we derived the following equation:

$$J_y P W_y - H_s W_x = M_y \tag{14-13}$$

substituting (14-9), (14-10), and (14-12) into (14-13):

$$J_y P(W_{OA} + P\alpha) - H_s(W_{IA} + \alpha W_{SRA}) = -K\alpha - CP\alpha + M_{tg} \tag{14-14}$$

which gives (expressed in the ge frame):

$$J_yP^2\alpha = H_sW_{IA} + M_{tg} - K\alpha - CP\alpha + H_s\alpha W_{SRA} - J_yPW_{OA}\,. \tag{14-15}$$

For the operational environment in which SDOF gyros are used, (αW_{SRA}) and PW_{OA} are negligible and, therefore, the last two terms on the RHS of (14-15) may be ignored. It follows from (14-15) :

$$J_yP^2\alpha = H_sW_{IA} - K\alpha - CP\alpha + M_{tg}. \tag{14-16}$$

Equation (14-16) shows that, in the absence of the commanded torque M_{tg}, the gyroscopic torque H_sW_{IA} causes angular acceleration $P^2\alpha$, which overcomes the elastic torque $K\alpha$ and the damping torque $CP\alpha$.

14.1 RATE GYROS

For rate gyro applications, $M_{tg} = 0$. From (14-16):

$$\frac{J_y}{K}P^2\alpha + \frac{C}{K}P\alpha + \alpha = \frac{H_s}{K}W_{IA} \tag{14-17}$$

By design $K >> C >> J_y$ for a rate gyro. So, for a constant input value of W_{IA}, α settles rather quickly to the final value of

$$\alpha = \frac{H_s}{K}W_{IA} \tag{14-18}$$

because $\frac{J_y}{K} << \frac{C}{K} << 1$ in (17).

Since the output angle α is proportional to the angular rate W_{IA} of the gyro unit (case) relative to the inertial space, the SDOF gyro of this type is called "rate gyro."

14.2 RATE INTEGRATING GYRO (RIG)

For a RIG, the elastic constant is removed, thus making $K = 0$, and the damping constant C is made large (larger than that of the rate gyro).

Thus, from (14-16) setting $K = 0$,

$$J_yP^2\alpha + CP\alpha = H_sW_{IA} + M_{tg} \tag{14-19}$$

or

$$P\left(\frac{J_y}{C}P\alpha + \alpha\right) = \frac{H_s}{C}W_{IA} + \frac{1}{C}M_{tg}. \tag{14-20}$$

For a constant input value of W_{IA}, the $\frac{J_y}{C}P\alpha$ term vanishes rather quickly (relative to α) because $C \gg J_y$ by design or $\frac{J_y}{C} << 1$, and (14-20) reduces to (in the absence of M_{tg}),

$$P\alpha = \frac{H_s}{C}W_{IA}. \tag{14-21}$$

Since $P(\cdot) = \frac{d}{dt}(\cdot)$, we have from (14-21)

$$\alpha = \frac{H_s}{C}\int_0^t W_{IA}\,dt. \tag{14-22}$$

According to (14-22), the angular displacement α of *ge* relative to *gu* (case) about the output axis is proportional to the integration with respect to time of the angular rate W_{IA}. Hence, the SDOF gyro of this type is called a "rate integrating gyro" or simply an "integrating gyro."

Denoting $\int_0^t W_{IA}\,d\tau = W_{IA}t = \theta_{IA}$ in (14-22) for a constant W_{IA}, we have

$$\alpha = \alpha_{0A} = \frac{H_s}{C}\theta_{IA} \tag{14-23}$$

It is to be noted that α_{OA} is not exactly equal to θ_{IA}, but scaled by $\frac{H_s}{C}$.

14.2.1 Gyro Transfer Function

(Those who are not familiar with the Laplace Transform method may skip this section)

By Laplace Transform, differential equations are transformed to algebraic equations. This sometimes facilitates the system analysis, without computer simulations of the differential equation in the time domain.

Denoting the Laplace Transform L of $\alpha(t)$ by $A(s)$, that is, $L[\alpha(t)] = A(s)$ and recalling that $L[P\alpha(t)] = SA(s)$ assuming $\alpha(0^+) = 0$

and

$L[P^2\alpha(t)] = s^2A(s)$ assuming $\alpha(0^+)=0$ $P\alpha(0^+)=0$, we get from (14-16), with $M_{tg} = 0$,

$$J_y s^2\alpha(s) + Cs\alpha(s) + K\alpha(s) = H_s W_{IA}(s) = (J_y S^2 + Cs + K)\alpha(s) = H_s W_{IA}(s) \quad (14\text{-}24)$$

which gives

$$\alpha(s) = \frac{H_s}{J_y s^2 + Cs + K} W_{IA}(s) \quad (14\text{-}25)$$

or

$$\alpha(s) = G(s)W_{IA}(s) \quad (14\text{-}26)$$

where

$$G(s) = \frac{\alpha_{OA}(s)}{W_{IA}(s)} = \frac{H_s}{J_y s^2 + Cs + K} \quad (14\text{-}27)$$

G(s) is called the transfer function, which relates the input $W_{IA}(s)$ to the output $\alpha(s) = \alpha_{OA}(s)$ with subscript OA indicating that α is the rotation about the output axis.

For a constant step input of W_{IA}, $W_{IA}(s) = \frac{1}{s} W_{IA}$.
Applying the final value theorem of $\lim_{t\to\infty} \alpha(t) = \lim_{s\to 0} s\alpha(s)$, we have, from (14-25),

$$\lim_{t\to\infty} \alpha(t) = \lim_{s\to 0} s \frac{H_s}{J_y s^2 + Cs + K} W_{IA} \quad (14\text{-}28)$$

or

$$\alpha(t) = \frac{H_s}{K} W_{IA} \quad \text{as } t \to \infty. \quad (14\text{-}29)$$

which is identical with (14-18) for the rate gyro. Recalling that $K >> C >> J_y$ for a rate gyro (that is, C is negligible relative to K, yet C/J_y is big), we can see from (14-28) that $\alpha(t)$ approaches the final value faster as *S* approaches zero than the case when *C* is comparable to *K* or bigger than *K*.

For <u>the integrating gyro</u> *K = 0*, while keeping $C >> J_y$. For this case, from (14-25)

$$\alpha(s) = \frac{H_s}{S(JS+C)} W_{IA}(s). \quad (14\text{-}30)$$

The transfer function in (14-30) has a pole at the origin (s = 0), and therefore, we cannot apply the final value theorem.

For a constant step input of $W_{IA}, W_{IA}(s) = \frac{1}{S} W_{IA}$. So, from (14-30)

$$\alpha(s) = \frac{1}{S}\left[\frac{H_s}{(JS+C)}\frac{W_{IA}}{S}\right] = \frac{1}{S}F(s) \tag{14-31}$$

where

$$F(s) = \frac{H_s}{(JS+C)}\frac{W_{IA}}{S} . \tag{14-32}$$

By denoting the inverse Laplace Transform of F(s) by $L^{-1}F(s) = f(t)$

$$f(t) = L^{-1}\left[\frac{\frac{H_s}{J}W_{IA}}{\left[S-\left(-\frac{C}{J}\right)\right][S-0]}\right]. \tag{14-33}$$

It follows, using the mathematical table for Laplace Transform :

$$f(t) = \frac{H_s W_{IA}}{J}\left[\frac{1}{-\frac{C}{J}}\left(\exp\left(-\frac{C}{J}t\right)-1\right)\right] = \frac{H_s W_{IA}}{C}\left(1-\exp\left(-\frac{C}{J}t\right)\right). \tag{14-34}$$

In a RIG C>>J by design or $\frac{C}{J} >> 1$, so we may regard $\exp\left(-\frac{C}{J}t\right)$ to approach zero very quickly. It follows from (14-34) for a step input of W_{IA}:

$$f(t) = \frac{H_s W_{IA}}{C} . \tag{14-35}$$

Using a Laplace Transform identity in (14-31) with (14-35),

$$\alpha(t) = L^{-1}\alpha(s) = L^{-1}\frac{1}{S}F(s)$$

$$= \int_0^t f(\tau)d\tau$$

$$= \int_0^t \frac{H_s W_{IA}}{C} dT = \frac{H_s}{C} W_{IA} \int_0^t dT = \frac{H_s}{C} W_{IA} t$$

or

$$\alpha(t) = \frac{H_s}{C} \theta_{IA} \qquad (14\text{-}36)$$

where θ_{IA} is the angular rotation about IA caused by W_{IA} for the duration of t, or $\theta_{IA} = W_{IA}\,t$. Note that (14-36) is identical with (14-23), for the RIG.

15.0 STABLE PLATFORM (IMU PLATFORM)

The stable platform of an LRBM is a platform, which is kept aligned with the earth-centered I-frames by a gimballed servo-mechanical control system, which uses gyros to detect rotational motion of the stable platform. Three mutually orthogonal accelerometers are rigidly attached to the stable platform and are used to determine the position and velocity of the missile. The stable platform with the accelerometers and the associated electronics is called an inertial measurement unit IMU and is the central part of the missile guidance system.

First, we explain the single-axis stable platform before discussing the more general three-axes stable platform. Consider a conceptual single-axis stable platform (plate) with the body-fixed axes as shown in Figure 15-1.

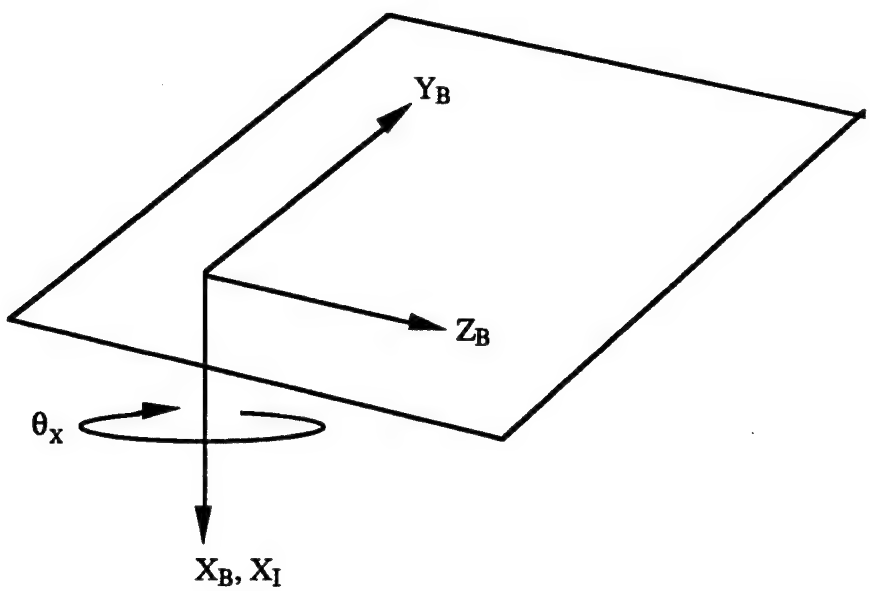

FIGURE 15-1. CONCEPTUAL SINGLE-AXIS STABLE PLATFORM

The X_B-axis is constrained to be parallel to thê X_I axis of the I-frame. The plate, which is perpendicular to the X_B-axis, is subject to the rotational disturbances about the X_B-axis caused by unwanted disturbances. This means that, although the plate itself has a fixed orientation in the inertial space, the Y_B-axis and Z_B-axis fixed on the plate may deviate from the original Y_I - axis and Z_I - axis.

The control problem is how to counterbalance the deviation of the Y_B-axis and Z_B-axis caused by the unwanted disturbance rotation about the X_B-axis, so that the Y_B-axis and Z_B-axis return to the directions parallel to the Y_I - axis and Z_I - axis.

This may be achieved by a scheme as shown in Figure 15-2. The center piece of the scheme is an SDOF gyro with the gyro unit's IA designated as the X-axis, the OA as the Y-axis, and the SRA as the Z-axis. SRA is the direction of the spin axis (SA) when SA is orthogonal to IA. SRA is the reference direction of SA. SA is always orthogonal to OA.

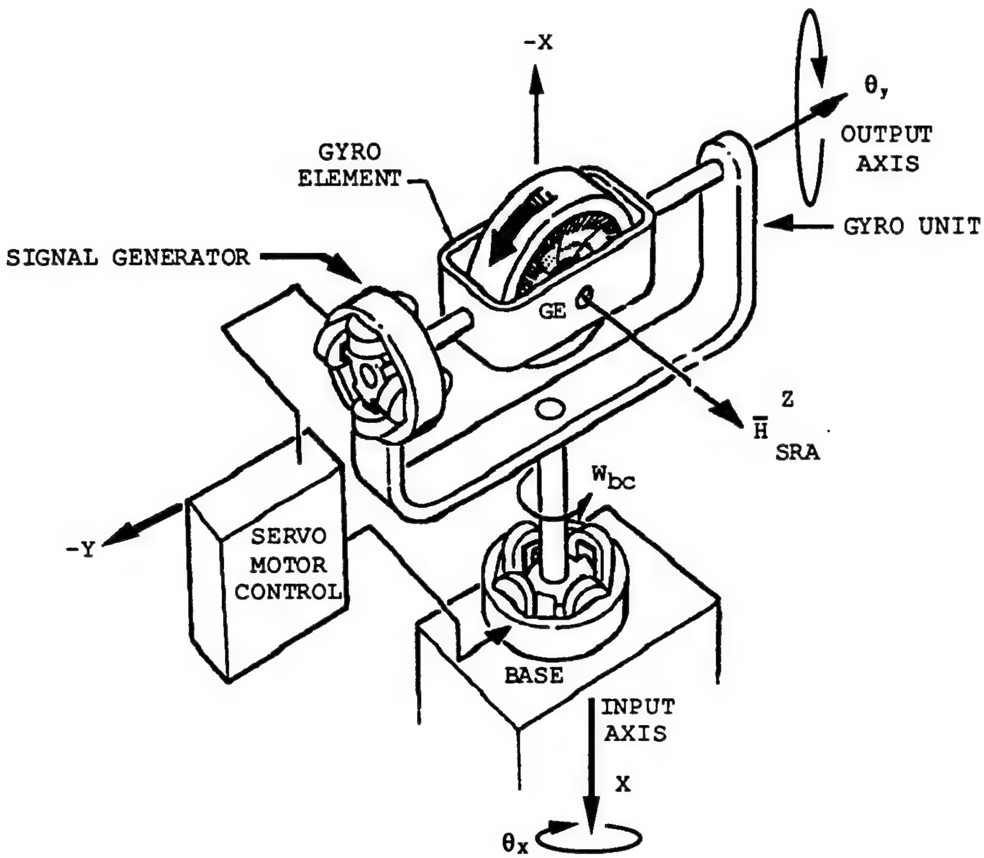

FIGURE 15-2. MECHANICAL SCHEME FOR SINGLE-AXIS STABLE PLATFORM

The type of gyro used for this purpose is called a RIG or simply an integrating gyro for the following reasons:

The IA of the gyro unit is constrained by the motor-shaft structure not to be rotatable with respect to the base except by the servo-motor control. When the base with the gyro on it (with the servo control off) rotates about the +X axis (IA) by an angle θ_x, the SA will precess (turn) downward toward the +X direction, following the gyroscopic principle that the spin vector will try to align itself with the torque vector (the direction of rotation).

This precession of SA rotates the gyro-element's SA axis by an angle θ_y (from SRA direction) about the OA (+Y direction).

For the RIG, by special design construction, θ_y is related to θ_x with close approximation for small angles by: (see a previous section on SDFG)

$$\theta_y = \frac{H_s}{C}\theta_x \tag{15-1}$$

in which H_s is the constant spin angular momentum of the gyro spin rotor, and C is the dynamic friction (so called damping) coefficient of the gyro element about the output axis (OA). Equation 15-1 shows that the output angle θ_y about *OA* is proportional to θ_x or to the integration of the angular rate $\dot{\theta}_x$ about *IA* which is the input angular rate. For this reason, the device is called a RIG.

The θ_y about *OA* caused by θ_x about *IA* is sensed by the signal generator (SG) mounted on the output-axis. This electrical signal from *SG* drives the servo-motor whose shaft coincides with the *IA* of the gyro, so that the gyro unit is rotated in the opposite direction from the initial θ_x rotation.

This operation turns the spin axis (SA of gyro) upward until θ_y becomes zero or θ_y is rebalanced, at which time the signal from *SG* becomes zero, causing the motor operation about *IA* to cease. With this control scheme, Y_B and Z_B axis misoriented by the initial rotation θ_x about IA of the platform is returned to the original Y_I and Z_I axes by the counter-rotation $(-\theta_x)$ about the *(+IA)* direction of the platform.

Referring to Figure 14.2, suppose the SA drifts downwards generating θ_y in the absence of rotation about the input axis (X_B-axis), even though Y_B and Z_B axis point to the correct directions, and thus do not need correction.

Nonetheless, in the actual system, the SG will detect θ_y and make the servo-motor turn the platform about *IA* in such a way as to null the θ_y. This would cause the Y_B and Z_B axes to deviate from the correct directions when correction is not necessary. This results in the platform drift-error caused by undesired drift of the gyro spin axis. This phenomenon is called gyro drift. By custom, the angular rate of gyro drift is simply called gyrodrift.

Consider a hypothetical platform (block) with three single-axis platform control systems with each SDOF gyro's *IA* , *OA,* and *SRA* oriented as shown in Figure 15.3.

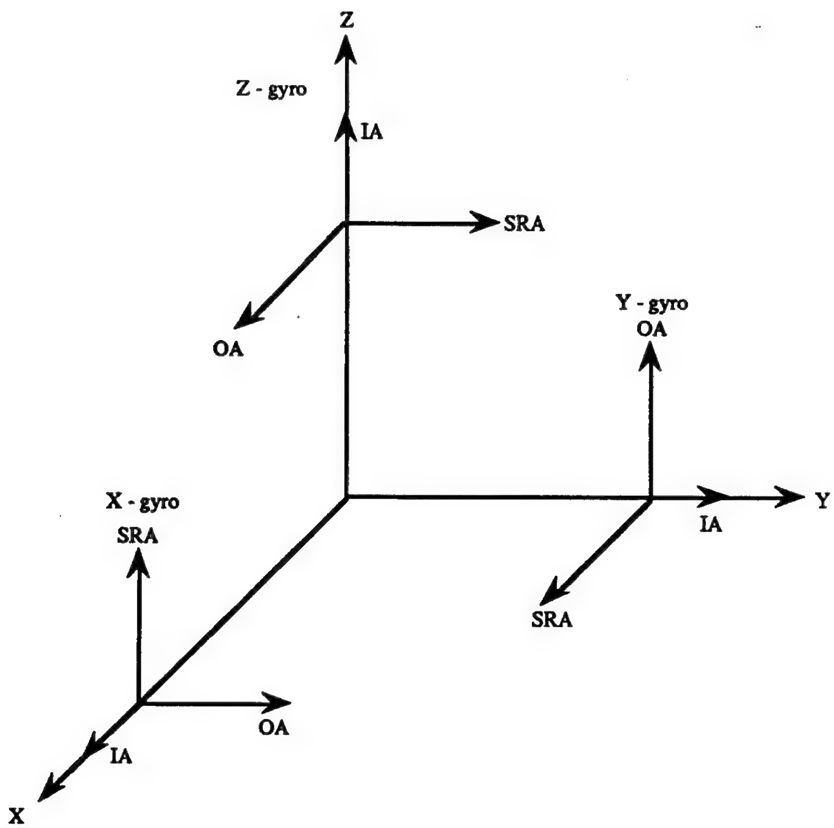

FIGURE 15-.3. THREE-AXIS STABLE PLATFORM

This is a hypothetical arrangement for tutorial purposes.

At t = 0, we assume the X_B, Y_B, Z_B axes of the platform are parallel to the X_I, Y_I, Z_I axes of an inertially nonrotating frame.

The input axis of the X-gyro designated by IA_x points to the inertially fixed direction if there are no rotations about IA_y (IA of Y-gyro) and IA_z (IA of Z-gyro). Similarly, IA_y points to the inertially fixed directions if there are no rotations about IA_x and IA_z, and IA_z points to the inertially fixed directions if there are no rotations about IA_x and IA_y. Thus, if, at $t = 0$ when IA_x, IA_y, and IA_z are parallel to X_I, Y_I, and Z_I axes, and if all three control systems function propertly, the platform would be inertially nonrotating about each of three axis and the platform becomes an inertially stablized platform, even under unwanted disturbances.

The actual three-axis stable platform control system is much more complicated. For those who are interested in the further study are referred to the following:

Inertial Navigation Systems
by: Charles Broxmeyer
McGraw-Hill, 1964

Gyroscopic Theory, Design and Instrumentation
by: Wrigley, Hollister and Denhard
The MIT Press, 1969

Control System Design Analysis of Three-Axis Dynamic Rate Table
by: Kee Soon Chun
MIT Dept. of Aeronautics and Astronautics
(SM Thesis), 1968

16.0 PENDULOUS INTEGRATING GYRO ACCELEROMETER (PIGA)

Consider an SDOF RIG with a mass unbalance *m* located at a distance *r* along the SA from the center of the gyro (the intersection of OA and SA) as shown in Figure 16-1.

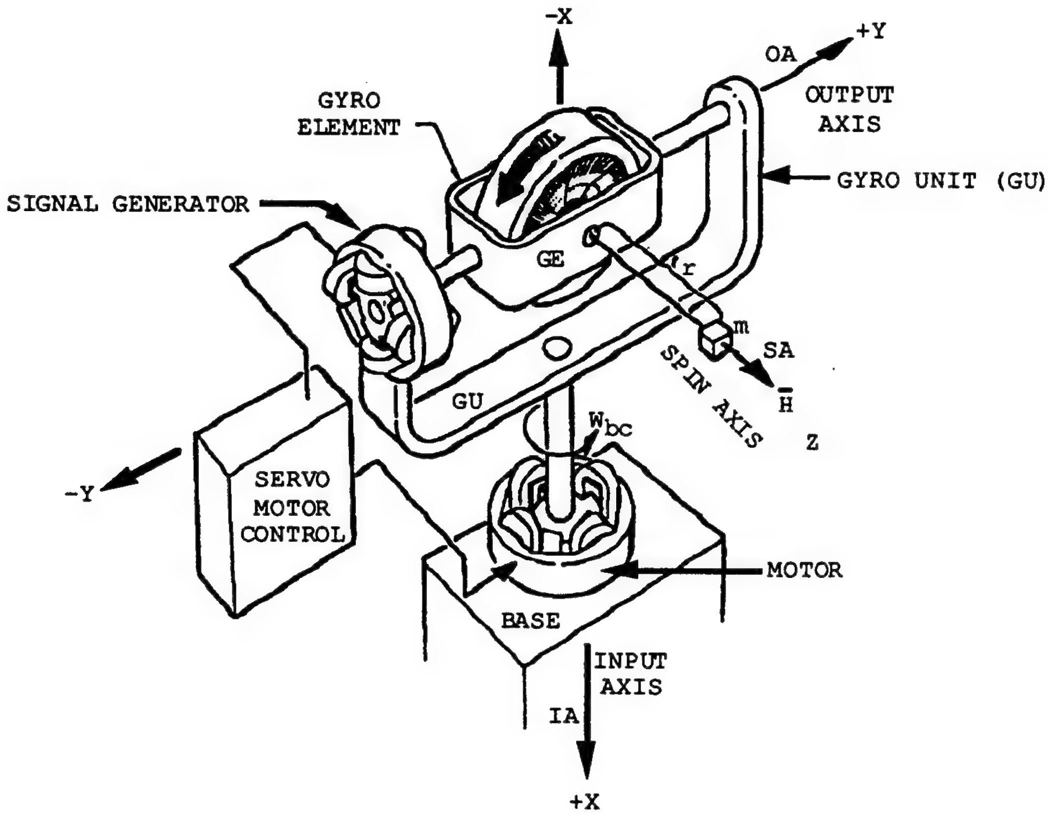

FIGURE 16-1. SDOF RIG WITH A MASS UNBALANCE *m*

The motor-shaft axis coincides with the IA of the gu. The motor is fixed to the base of an inertially nonrotating platform.

Suppose the platform, such as an IM of the missile, is under acceleration whose nongravitational portion (specific force f_{NG}) is along the -X direction. This causes the mass *m* to experience the inertial reaction force toward +X direction, generating a torque about OA, that is, about the +Y axis. If the gyro unit were free to rotate about the gyro IA, the spin vector H would rotate (precess) in the Y - Z plane about the -X axis (W_{bc}), trying to align the H-vector with the applied torque (vector) about the +Y-axis. The precession of H-vector is caused by the inertial reaction force on *m*, according to the gyroscopic principle.

However, GU (whose input axis coincides with the motor shaft) is prevented from rotating about its IA by the fixed motor structure. Therefore, the SA with the mass-unbalance on it has no choice except to move downwards, causing the SA to rotate about OA (+Y axis) creating an angle *A* from the original null position. This rotation is sensed by SG and drives the motor shaft through the servo motor control in such a way that the angle *A* is nulled (rebalanced)—in this case by rotating the motor shaft in the -X direction. When *A* reduces to zero, the output of the SG correspondingly reduces to zero as well, and the motor shaft rotation stops.

For the reasons explained in the previous sections on accelerometers, f_G (gravitational specific force) does not contribute to the generation of torque. Only f_{NG} causes the reaction-specific force f_R, which generates the torque. The torque generated by the inertial reaction force because of f_{NG} on the mass *m* located at *r* on the SA is $mrf_R = -mrf_{NG}$ with the negative sign indicating that f_R is in the opposite direction from f_{NG}.

From Equation (16-19) from the Section 13.0 on the SDOF gyro,

$$J_{OA}P2A_{OA} + CpA_{OA} = H_sW_{IA} + mrf_{NG}(1)$$

In (16-1), the applied torque mrf_{NG} is nulled (rebalanced) by the opposing gyroscopic torque H_SW_{IA} caused by the rotation of GU about the negative input axis (about -X axis) by the servo motor, thus causing the direction of the torque generated by H_SW_{IA} to point in the -Y direction, which is opposite the direction (+Y direction) of the torque generated by mrf_{NG}.

Under steady-state conditions, both $PA_{OA} = \frac{d}{dt}A_{OA}$ and $P^2A_{OA} = \frac{d^2}{dt^2}A_{OA}$ are zero. It follows from (16-1):

$$H_sW_{IA} = -mrf_{NG}$$

Writing $W_{IA} = \frac{d\theta_{IA}}{dt}$, $f_{NG} = \frac{dV_{NG}}{dt}$ and integrating both sides from 0 to T, the computation period, we have

$$\int_0^T \frac{d\theta_{IA}}{dt}\,dt = -\frac{mr}{H}\int_0^T \frac{dV_{NG}}{dt}\,dt$$

or

$$\theta_{IA} = -\frac{mr}{H}V_{NG}$$

That is, the angular rotation θ about IA accumulated during time *T* is proportional (scaled by $\frac{mr}{H}$) to the velocity accumulated (during *T*) as a result of the nongravitational specific force f_{NG} applied in the -X direction.

This device senses and measures that part of the velocity, which is the integration of specific force (acceleration) (caused by the nongravitational specific force f_{NG}) by means of a gyro with a pendulous mass deliberately placed on the SA. Hence, it is called a pendulous integrating gyro accelerometer or PIGA.

As we have seen before in the section on SDOF RIG (Equation 16-23 of Section 14.0), the A_{0A} of *GE* from *GU* is related to θ_{IA} by

$$A_{OA} = \frac{H_s}{C}\theta_{IA}$$

The A_{0A} is measurable by the signal generator attached to 0A. Thus, in practice, V_{NG} may be determined by the summation of the electric pulses from the signal generator attached on 0A, rather than by the mechanical summation of θ_{IA}.

Unlike PIPA, PIGA does not suffer from the voltage and current magnitude variations. Also, PIGA is almost insensitive to the variations of the magnetic flux density and permeability. This explains why PIGA replaced PIPA in the LRBM.

17.0 EQUATIONS OF MOTION FOR INERTIAL NAVIGATION

By Newton's second law:

$$mP_I^2R_{EP} = F_G + F_{NG} \tag{17-1}$$

Since the external forces have to be either gravitational force (F_G) or nongravitational force (F_{NG}) in a mutually exclusive way, by denoting

$f_G = \frac{F_G}{m}$ and $f_{NG} = \frac{F_{NG}}{m}$, we may express (17-1) by:

$$P_I^2R_{EP} = \frac{F_G}{m} + \frac{F_{NG}}{m} = f_G + f_{NG} \tag{17-2}$$

Equation (17-2) is the equation of motion to implement an INS in the earth-centered I-frame. We have previously found that the onboard accelerometer senses nongravitational specific force f_{NG} only, leaving out the gravitational specific force, f_G. This necessitates the determination of f_G (as a function of position) by the onboard computer, based on a mathematical model to implement Newton's second law by onboard instruments and a computer in a self-contained way without external help.

The block diagram for an INS implementation in an I-frame is shown in Figure 17-1 below.

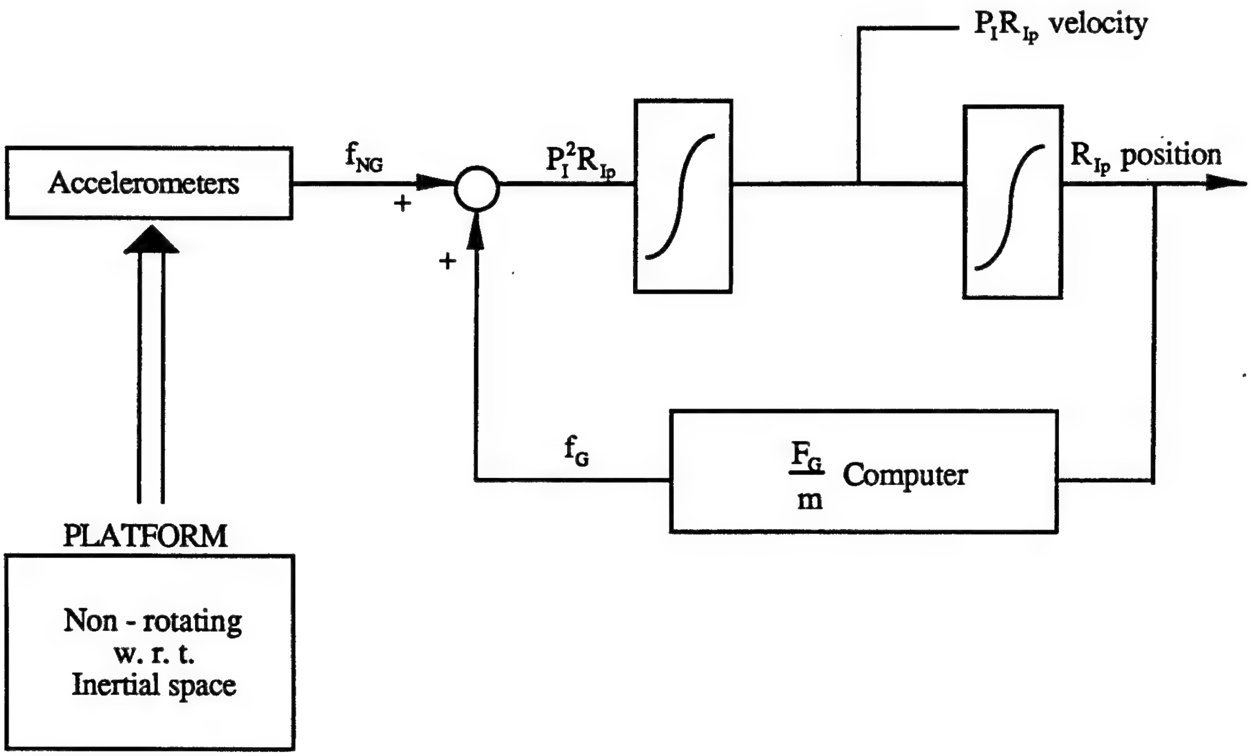

FIGURE 17-1. INS IMPLEMENTATION IN AN I-FRAME

Now, to derive the equation of motion in the earth-fixed frame (that rotates with the earth WRT the I-frame with angular velocity W_{IE}), we apply the theorem of Coriolis to R_{Ep} between the earth-centered I-frame and the earth-fixed (and earth-centered) E-frame.
Thus,

$$P_I R_{Ep} = P_E R_{Ep} + W_{IE} \times R_{Ep} \, . \tag{17-3}$$

It follows

$$P_I^2 R_{Ep} = P_I\left(P_I R_{Ep}\right)$$

$$= P_I\left(P_E R_{Ep} + W_{IE} \times R_{Ep}\right) \quad \text{using (17-3)}$$

$$= P_I\left(P_E R_{Ep}\right) + P_I\left(W_{IE} \times R_{Ep}\right) . \tag{17-4}$$

Referring to the right-hand side of (17-4), by applying the theorem of Coriolis to the first term:

$$P_I\left(P_E R_{Ep}\right) = P_E\left(P_E R_{Ep}\right) + W_{IE} \times \left(P_E R_{Ep}\right)$$

$$= P_E^2 R_{Ep} + W_{IE} \times \left(P_E R_{Ep}\right) . \tag{17-5}$$

Noting that W_{IE} is constant and therefore $P_I W_{IE} = 0$, we have, from the second term on the right-hand side of (17-4)

$$P_I\left(W_{IE} \times R_{Ep}\right) = \left(P_I W_{IE}\right) \times R_{Ep} + W_{IE} \times \left(P_I R_{Ep}\right)$$

$$= W_{IE} \times \left(P_I R_{EP}\right)$$

$$= W_{IE} \times \left(P_E R_{EP} + W_{IE} \times R_{EP}\right)$$

$$= W_{IE} \times \left(P_E R_{Ep}\right) + W_{IE} \times \left(W_{IE} \times R_{Ep}\right) . \tag{17-6}$$

Substituting (17-2), (17-5) and (17-6) into (17-4):

$$f_G + f_{NG} = P_E^2 R_{Ep} + 2 W_{IE} \times \left(P_E R_{Ep}\right) + W_{IE} \times \left(W_{IE} \times R_{Ep}\right) . \tag{17-7}$$

For the case in which the point P is stationary on earth, R_{Ep}^{E} is constant and, therefore,

$P_E R_{EP} = 0$ and therefore $P_E\left(P_E R_{EP}\right) = P_E^2 R_{Ep} = 0.$

For this special case, (17-7) reduces to

$$f_G - W_{IE} \times \left(W_{IE} \times R_{Ep}\right) = -f_{NG} . \tag{17-8}$$

As we have previously discussed in the section entitled *The Prototype Linear Accelerometer*, the accelerometer output measures $-f_{NG}$. Thus, the left-hand side of (17-8) or f_G -

$W_{IE} \times (W_{IE} \times R_{EP})$ is the vector quantity (both direction and magnitude), which a plum-bob on a simple pendulum at rest on earth senses. We call this vector quantity "gravity" and denote it by g. That is,

$$g \,\Delta\, f_G - W_{IE} \times \left(W_{IE} \times R_{Ep}\right). \tag{17-9}$$

Note that in (17-9) $g = f_G$ for $W_{IE} = 0$ for the nonrotating earth. For the rotating earth, g is not equal to f_g except at the north and south poles, where $W_{IE} \times R_{Ep} = 0$.

Substituting (17-9) into (17-7):

$$f_{NG} + g = P_E^2 R_{Ep} + 2W_{IE} \times \left(P_E R_{Ep}\right). \tag{17-10}$$

Applying the theorem of Coriolis to $(P_E R_{EP})$ between the E-frame and N-frame where the N-frame is the local-level northeast down frame,

$$P_E^2 R_{Ep} = P_E\left(P_E R_{Ep}\right)$$

$$= P_N\left(P_E R_{Ep}\right) + W_{EN} \times \left(P_E R_{Ep}\right). \tag{17-11}$$

Substituting (17-11) into (17-10), and coordinating (expressing vector-components) in the N-frame:

$$\left[P_N\left(P_E R_{Ep}\right)\right]^N = g^N + f_{NG}^N - \left(2W_{IE}^N + W_{EN}^N\right) \times \left(P_E R_{Ep}\right)^N. \tag{17-12}$$

Equation (17-12) is the equation of motion for the near-earth (land, air, and sea) navigation implementing a local-level (north, east, and down (NED) platform. It implies that the velocity relative to the earth, $P_E R_{Ep}$ and the position relative to the earth, R_{Ep} (which is the integration of $P_E R_{EP}$) may be determined based on the f_{NG}^N, which is the negatives of the outputs of three accelerometers along the NED directions on the platform, which implement the N-frame.

Referring to left-hand side (LHS) of (17-12), we see that

$$\left[P_N\left(P_E R_{EP}\right)\right]^N = P_N\left(P_E R_{EP}\right)^N$$

$$= P\left(P_E P_{EP}\right)^N$$

Therefore, since $p(\cdot) = \frac{d}{dt}(\cdot)$,

$$\int \left[P \left(P_E R_{EP} \right)^N \right] dt = \int \frac{d}{dt} \left(P_E R_{EP} \right)^N dt$$

$$= \left(P_E R_{EP} \right)^N = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \tag{17-13}$$

where V_x, V_y, and V_z are respectively NED velocities of the vehicle relative to the earth-fixed frame.

The vehicle's position relative to the earth (latitude and longitude) is determined from the velocity output by integration in the following way.

For the spherical earth, denoting latitude by ϕ and longitude by λ, and the magnitude of R_{Ep} by R, we get (refer to Figure 17-2):

$V_N = \dot{\phi} R$ or $\dot{\phi} = \frac{V_N}{R}$ so that

$$\int \frac{V_N}{R} dt = \int \dot{\phi}\, dt \tag{17-14}$$

$$= \phi$$

$$= \text{latitude}$$

and $V_E = (R\cos\phi)\dot{\lambda}$ or $\dot{\lambda} = \frac{V_E}{R\cos\phi}$

so that

$$\int \frac{V_E}{R\cos\phi} dt = \int \dot{\lambda} dt = \lambda = \text{longitude.} \tag{17-15}$$

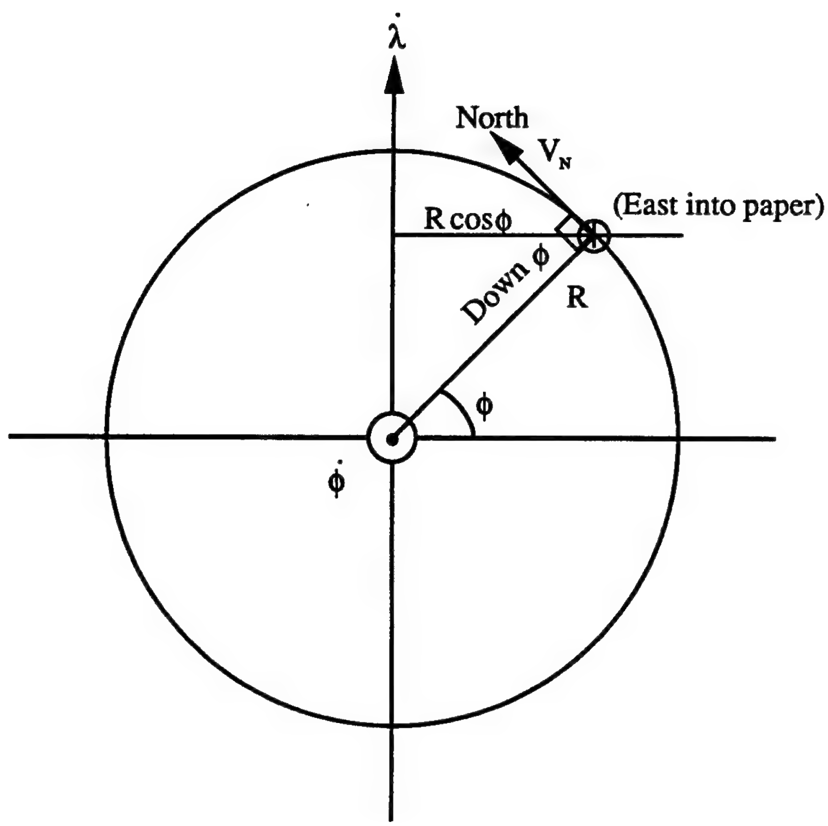

FIGURE 17-2. DETERMINATION OF LINEAR VELOCITIES FROM THE LATITUDE AND LONGITUDE RATES

Next, by denoting the magnitude of W_{IE} by Ω, we have (refer to Figure 17-3):

$$W_{IE}^{N} = \begin{pmatrix} W_{IEx} \\ W_{IEy} \\ W_{IEz} \end{pmatrix} = \begin{pmatrix} \Omega\cos\phi \\ 0 \\ -\Omega\sin\phi \end{pmatrix}. \tag{17-16}$$

The negative sign in W_{IEZ} is necessary because the vertical component of W_{IE} is upward and, therefore, opposite to the Z - direction, which is downward.

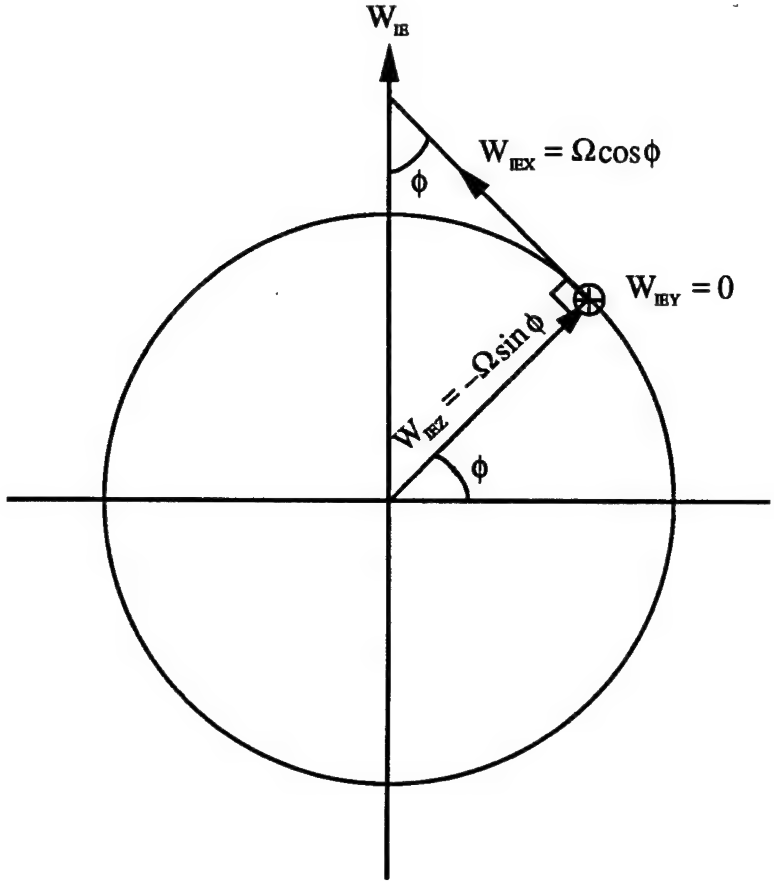

FIGURE 17-3. DECOMPOSITION OF THE EARTH RATE INTO NED COMPONENTS

For W^N_{EN} (angular velocity of the local-level NED frame relative to the earth-fixed frame), the N-frame rotates relative to the E-frame caused either by latitude rate $\dot{\phi}$ or longitude rate $\dot{\lambda}$. Referring to Figure 17-4, the North component is $\dot{\lambda}\cos\phi$ and the Z (down) component is $-\dot{\lambda}\sin\phi$, because $\dot{\lambda}\sin\phi$ is in the upward direction, which is opposite the positive Z (downward) direction. The north and down components are not influenced by $\dot{\phi}$ because $\dot{\phi}$ is perpendicular to the meridian plane. The east direction is opposite to the $\dot{\phi}$ direction (west). Therefore, the east component $-\dot{\phi}$ is not influenced by $\dot{\lambda}$ because $\dot{\phi}$ is perpendicular to $\dot{\lambda}$. Thus,

$$W^N_{EN} = \begin{pmatrix} W_{ENX} \\ W_{ENY} \\ W_{ENZ} \end{pmatrix} = \begin{pmatrix} \dot{\lambda}\cos\phi \\ -\dot{\phi} \\ -\dot{\lambda}\sin\phi \end{pmatrix} \tag{17-17}$$

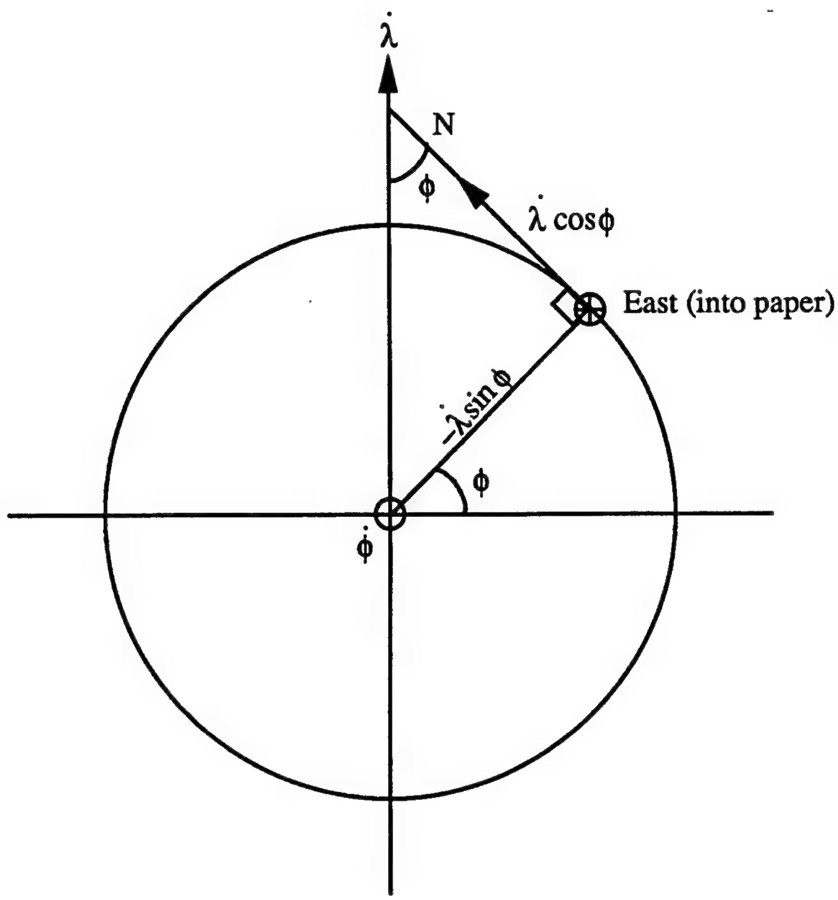

FIGURE 17-4. DECOMPOSITION OF THE LONGITUDE RATE INTO NORTH AND DOWN COMPONENTS

To maintain the platform (on which NED accelerometers are mounted) in the local-level (NED) frame, it has to be torqued by the angular rate W^N_{IN}.

Using (17-16) and (17-17):

$$W^N_{IN} = W^N_{IE} + W^N_{EN}$$

$$= \begin{pmatrix} \Omega\cos\phi \\ 0 \\ -\Omega\sin\phi \end{pmatrix} + \begin{pmatrix} \dot{\lambda}\cos\phi \\ -\dot{\phi} \\ -\dot{\lambda}\sin\phi \end{pmatrix}$$

$$=\begin{pmatrix} \left(\Omega+\dot{\lambda}\right)\cos\phi \\ -\dot{\phi} \\ -\left(\Omega+\dot{\lambda}\right)\sin\phi \end{pmatrix}. \qquad (17\text{-}18)$$

The implementation of Equation (17-12) for terrestrial navigation using the local-level (NED) frame is shown in Figures 17-5 and 17-6.

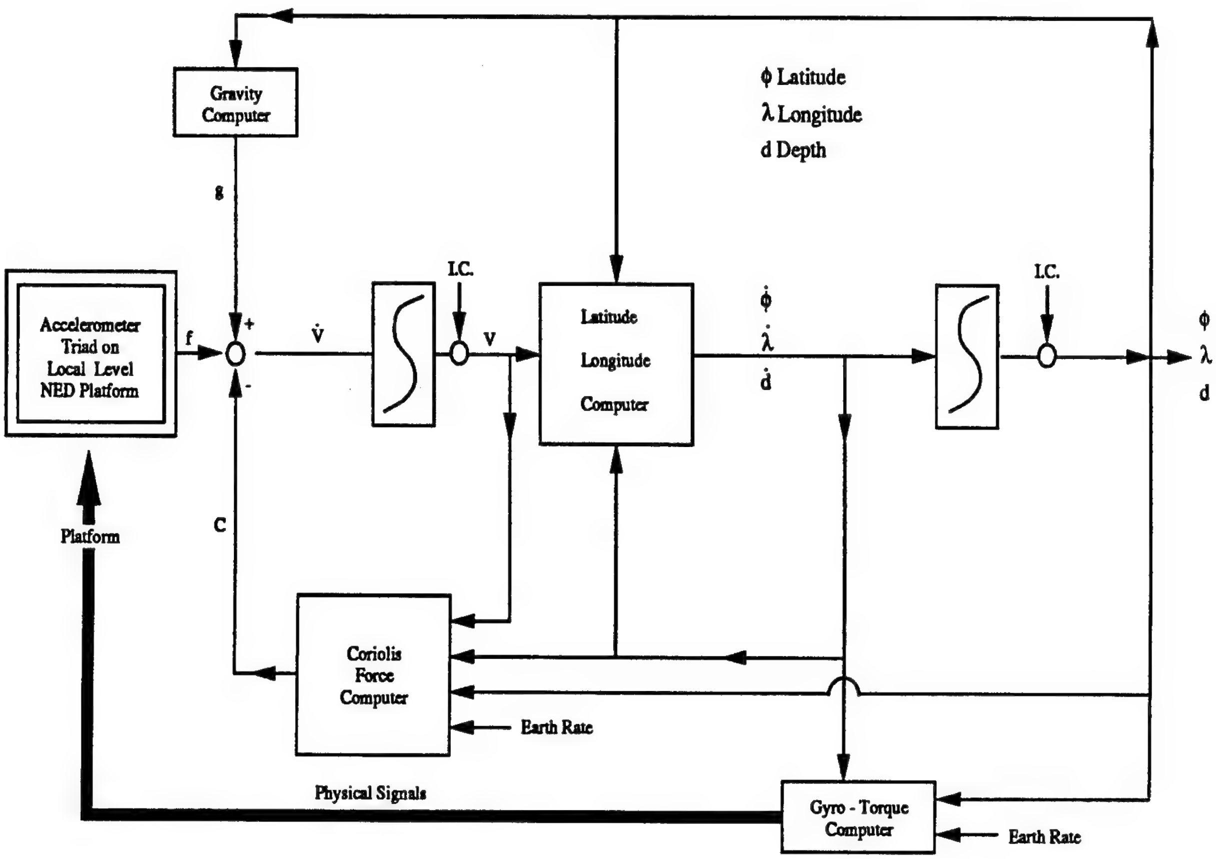

FIGURE 17-5. SIGNAL FLOW BLOCK DIAGRAM OF LOCAL-LEVEL N-FRAME IMPLEMENTATION OF TERRESTRIAL NAVIGATION

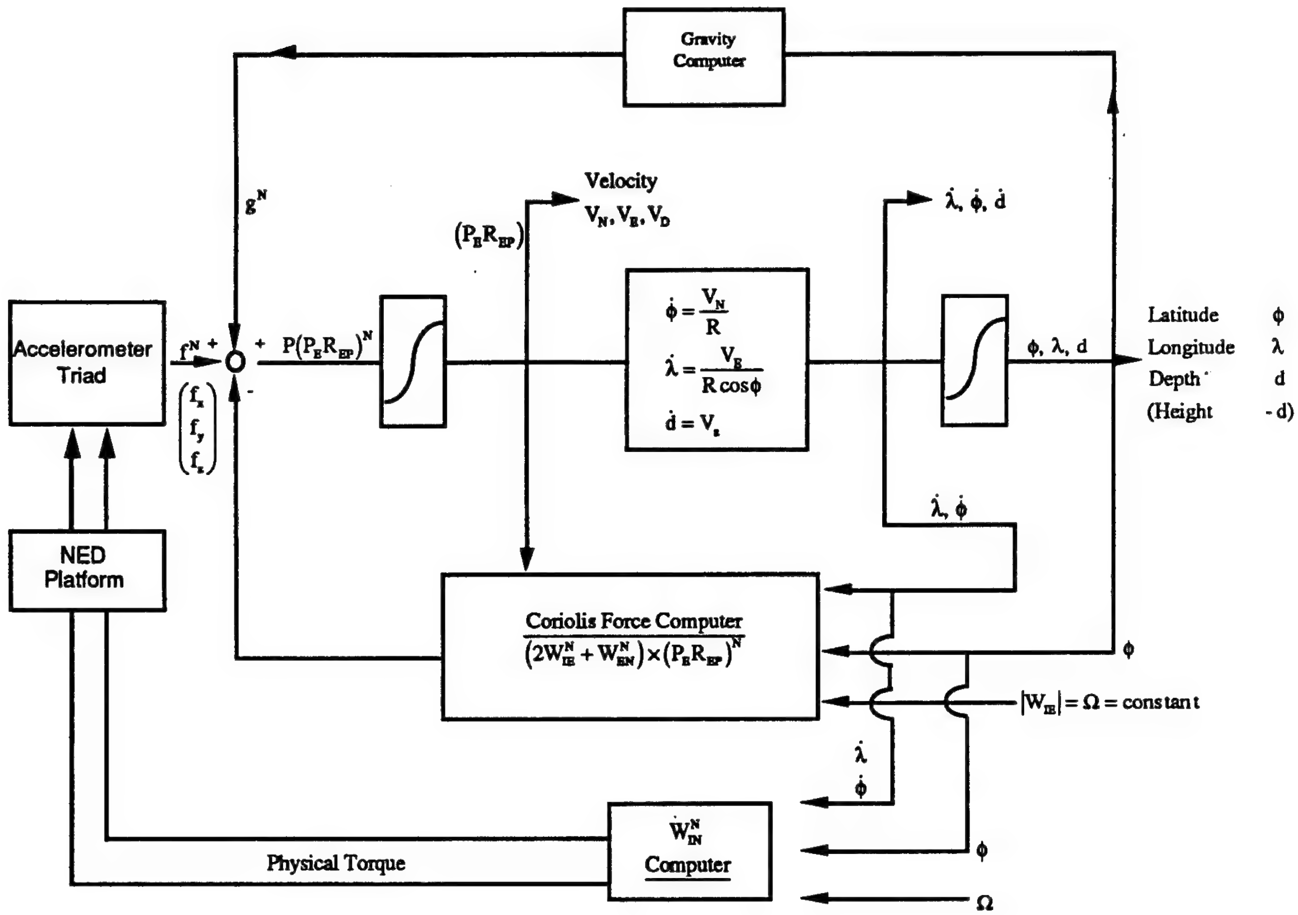

FIGURE 17-6. IMPLEMENTATION OF NAVIGATION EQUATION (17-12)

Repeating Equation (17-12) :

$$\left[P_N\left(P_E R_{Ep}\right)\right]^N = g^N + f^N_{NG} - \left(2W^N_{IE} + W^N_{EN}\right) \times \left(P_E R_{Ep}\right)^N \qquad (17\text{-}12)$$

where

$$\left(P_E R_{EP}\right)^N = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_N \\ V_E \\ V_D \end{pmatrix}$$

$$[P_N(P_E R_{EP})]^N = P\left(P_E R_{EP}{}^N\right) = \begin{pmatrix} \dot{V}_x \\ \dot{V}_y \\ \dot{V}_z \end{pmatrix} = \begin{pmatrix} \dot{V}_N \\ \dot{V}_E \\ \dot{V}_D \end{pmatrix}$$

$$f^N = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \qquad g^N = \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix}$$

$$W_{IE}^N = \begin{pmatrix} W_{IEx} \\ W_{IEy} \\ W_{IEz} \end{pmatrix} \qquad W_{EN}^N = \begin{pmatrix} W_{ENx} \\ W_{ENy} \\ W_{IEz} \end{pmatrix}$$

then

$$\left(2W_{IE}^N + W_{EN}^N\right) \times \left(P_E R_{Ep}\right)^N$$

$$= \begin{vmatrix} i & j & k \\ 2W_{IEx} + W_{ENx} & 2W_{IEy} + W_{ENy} & 2W_{IEz} + W_{ENz} \\ V_x & V_y & V_z \end{vmatrix} \text{i, j, k of N - frame}$$

$$= \begin{pmatrix} \left(2W_{IEy} + W_{ENy}\right)V_z - \left(2W_{IEz} + W_{ENz}\right)V_y \\ \left(2W_{IEz} + W_{ENz}\right)V_x - \left(2W_{IEx} + W_{ENx}\right)V_z \\ \left(2W_{IEx} + W_{ENx}\right)V_y - \left(2W_{IEy} + W_{ENy}\right)V_x \end{pmatrix}$$

substituting these into (17-12):

$$\dot{V}_x = f_x + g_x + \left(2W_{IEz} + W_{ENz}\right)V_y - \left(2W_{IEy} + W_{ENy}\right)V_z$$

$$\dot{V}_y = f_y + g_y + \left(2W_{IEx} + W_{ENx}\right)V_z - \left(2W_{IEz} + W_{ENz}\right)V_x$$

$$\dot{V}_z = f_z + g_z + (2W_{IEy} + W_{ENy})V_x - (2W_{IEx} + W_{ENx})V_y$$

in which, referring to (17-16) and (17-17),

$$W_{IEx} = \Omega\cos\phi$$

$$W_{IEy} = 0$$

$$W_{IEz} = -\Omega\sin\phi$$

$$W_{ENx} = \dot{\lambda}\cos\phi$$

$$W_{ENy} = -\dot{\phi}$$

$$W_{ENz} = -\dot{\lambda}\sin\phi$$

18.0 DETERMINATION OF INITIAL POSITION AND INITIAL VELOCITY

In inertial navigation, we determine the acceleration $P_I^2 R_{EP}$ from

$$P_I^2 R_{EP}(t) = f_G + f_{NG} \tag{18-1}$$

where

R_{EP} is the displacement vector from the center of the earth to the point of accelerometer location, and f_G is obtained from the mathematical model as a function of position and f_{NG} from the accelerometer measurements. The velocity $P_I R_{EP}$ is obtained by integrating $P_I^2 R_{EP}$ that is,

$$P_I R_{EP}(t) = \int_0^t P_I^2 R_{EP}(\tau) d\tau + P_I R_{EP}(0) \tag{18-2}$$

where $P_I R_{EP}(0)$ is $P_I R_{EP}$ at $t = 0$. The position R_{EP} is obtained by integrating $P_I R_{EP}$ that is,

$$R_{EP}(t) = \int_0^t P_I R_{EP}(\beta) d\beta + R_{EP}(0) \tag{18-3}$$

where $R_{EP}(0)$ is R_{EP} at $t = 0$.

Thus, we see that, to determine the current position $R_{EP}(t)$, we need the initial velocity $P_I R_{EP}(0)$ and the initial position $R_{EP}(0)$.

18.1 DETERMINATION OF THE INITIAL POSITION

We start with

$$R_{EP}^I = R_{ES}^I + R_{SP}^I \tag{18-4}$$

where

E = Center of the earth
S = Center of the submarine (ship or aircraft) INS
P = Center of the missile guidance system.

and the superscript I indicates that all components are expressed in the I-frame. In (18-4), the submarine's location R_{ES} is either available as an output from its navigation system or may be computed from it.

The position R^S_{SP} of the missile guidance system, before launch, while in the submarine or aircraft is given in the ship-fixed S-frame. The origin of the S-frame is at the center of the ship navigation system. The x-axis of the frame is along the longitudinal axis on the plane paralleled to the main (or conceptual) deck through the origin of the frame. The y-axis is perpendicular, on the plane, to the x-axis on the plane. The z-axis is perpendicular to both x and y axes according to the right-hand rule. To express R^S_{SP} as R^I_{SP}, we have to know C^I_S so that,

$$R^I_{SP} = C^I_S R^S_{SP} . \tag{18-5}$$

The coordinate transformation matrix C^I_S from the S-frame to the I-frame is determined from the gimbal angles of the submarine (or aircraft) INS (SINS) if it implements the I-frame. If it implements the navigation frame (NED), C^N_S is first determined from the gimbal angles of SINS, and then C^I_S is obtained by

$$C^I_S = C^I_N C^N_S . \tag{18-6}$$

The coordinate transformation matrix C^I_N from the N-frame to I-frame is given in terms of the submarine's latitude, ϕ (an output from the SINS), and the celestial longitude, λ_C, by (with the derivation to follow):

$$C^I_N = \begin{pmatrix} -\sin\phi\cos\lambda_c & -\sin\lambda_c & -\cos\phi\cos\lambda_c \\ -\sin\phi\sin\lambda_c & \cos\lambda_c & -\cos\phi\sin\lambda_c \\ \cos\phi & 0 & -\sin\phi \end{pmatrix} \tag{18-7}$$

where

$$\lambda_C = \lambda_G + \lambda$$

in which

λ_G = Celestial longitude of Greenwich meridian

λ = Terrestrial longitude of the submarine (relative to the Greenwich meridian)

λ_C = Celestial longitude of the submarine (relative to the inertially fixed reference)

18.2 DERIVATION OF C^I_N

This derivation is shown here in detail as a sample case of determination of coordinate transformation matrix for the tutorial purpose.

It turns out that it is easier to get C^N_I first, and then get C^I_N by transposing C^N_I or

$C_N^I = \left(C_I^N\right)^T.$

To get C_I^N (refer to Figures 18-1a and 18-1b), rotate I-frame (with X_I, Y_I, Z_I axes) about Z_I axis by an angle of λ_c (celestial longitude). Let us call the resulting frame the Λ–frame (with X_λ, Y_λ, Z_λ axes).

This operation yields (from Figures 18-1a and 18-1b) :

$$\begin{pmatrix} X_\lambda \\ Y_\lambda \\ Z_\lambda \end{pmatrix}^{\wedge} = \begin{pmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}_I^{\wedge} \begin{pmatrix} X_I \\ Y_I \\ Z_I \end{pmatrix}^I \tag{18-8}$$

or

$$R^{\wedge} = C_I^{\wedge} R^I \tag{18-9}$$

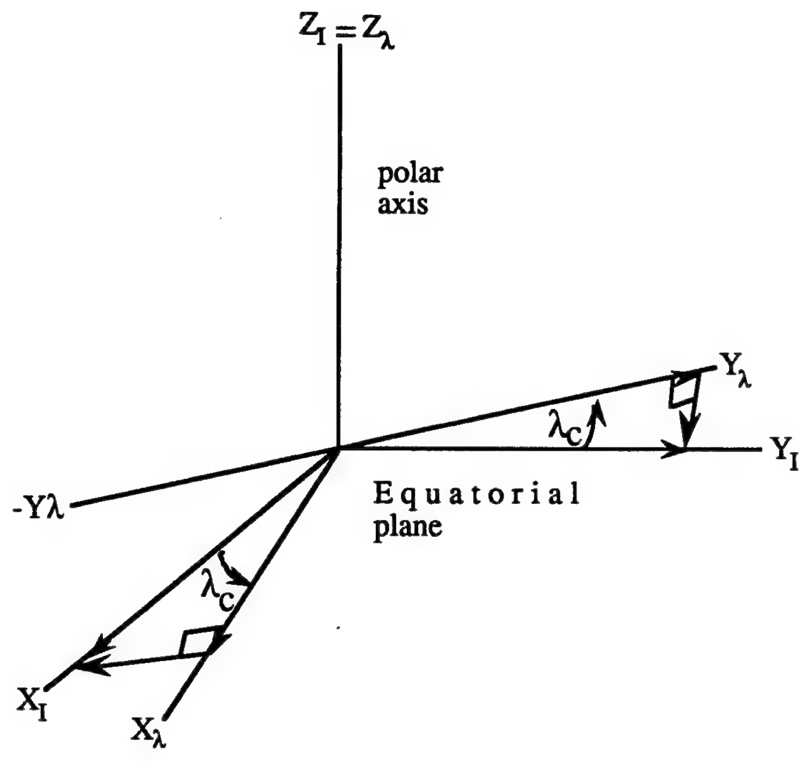

FIGURE 18-1A. DETERMINATION OF $C_I^{\wedge}$

$X_\lambda = X_I \cos\lambda_c + Y_I \sin\lambda_c$

$X_\lambda = -X_I \sin\lambda_c + Y_I \cos\lambda_c$

$Z_\lambda = Z_I$

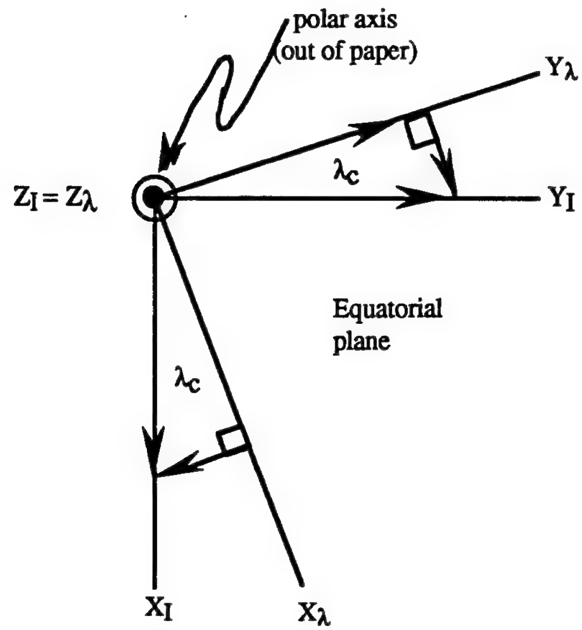

FIGURE 18-1B. DETERMINATION OF C_I^Λ

Next, rotate Λ - frame about the negative Y_λ axis by an angle of ϕ (latitude) as shown in Figure 18-2.

Let us call the resulting frame the Φ - frame (with X_ϕ, Y_ϕ and Z_ϕ axes).

This operation yields:

$$\begin{pmatrix} X_\phi \\ Y_\phi \\ Z_\phi \end{pmatrix}^{\Phi} = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}_{\Lambda}^{\Phi} \begin{pmatrix} X_\lambda \\ Y_\lambda \\ Z_\lambda \end{pmatrix}^{\Lambda} \tag{18-10}$$

or

$$R^\Phi = C_\Lambda^\Phi R^\Lambda \tag{18-11}$$

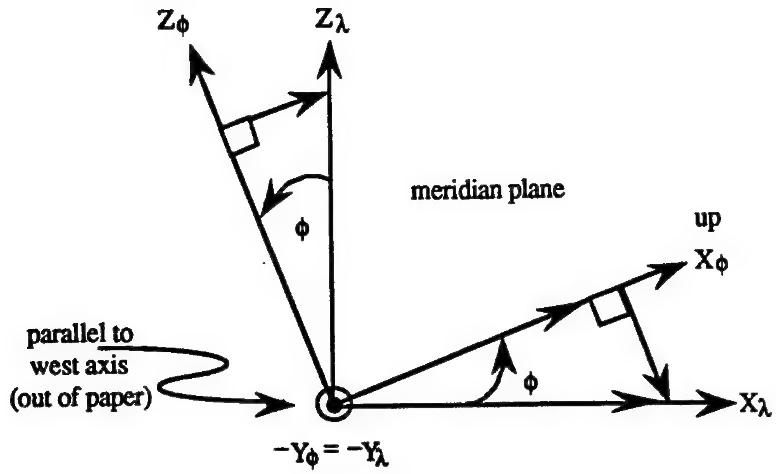

FIGURE 18-2A. DETERMINATION OF C_{Λ}^{ϕ}

$X_{\varphi} = X_{\lambda}\cos\varphi + Z_{\lambda}\sin\phi$

$-Y_{\varphi} = -Y_{\lambda}$ or $Y_{\varphi} = Y_{\lambda}$

$Z_{\varphi} = -X_{\lambda}\sin\varphi + Z_{\lambda}\cos\phi$

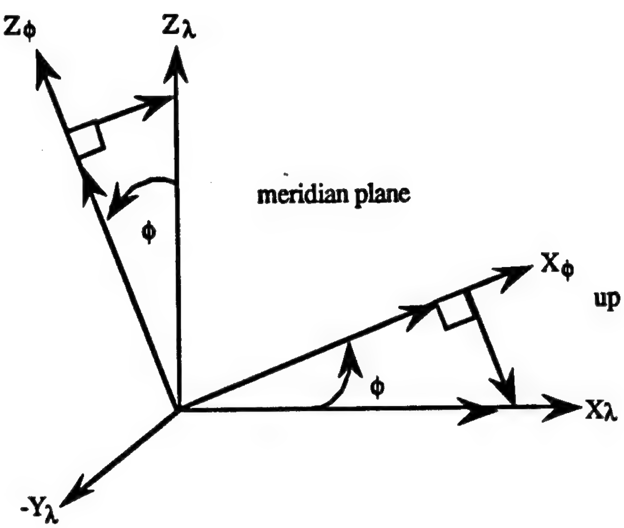

FIGURE 18-2B. DETERMINATION OF C_{Λ}^{ϕ}

Referring to Figure 18-3 below, we see that the N-frame (with X_N (north), Y_N (east), Z_N (down) axes) is related to Φ - frame (shifted from the center of the earth to a parallel frame located at the vehicle position on the meridian) by

$$X_N = Z_\phi \,; \quad Y_N = Y_\phi \,; \quad Z_N = X_\phi$$

or

$$\begin{pmatrix} X_N \\ Y_N \\ Z_N \end{pmatrix}^N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^N_\Phi \begin{pmatrix} X_\phi \\ Y_\phi \\ Z_\phi \end{pmatrix}^\Phi \tag{18-12}$$

or

$$R^N = C^N_\phi R^\phi \tag{18-13}$$

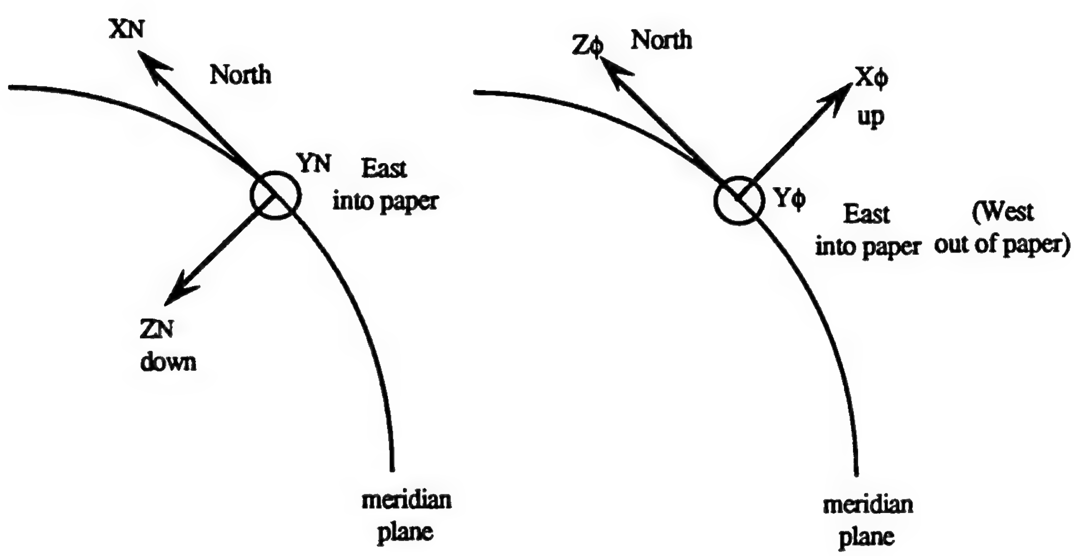

FIGURE 18-3. DETERMINATION OF C^N_ϕ

Thus, from (18-13), (18-11), and (18-9)

$$\begin{aligned} R^N &= C^N_\Phi R^\Phi \\ &= C^N_\Phi C^\Phi_\wedge R^\wedge \end{aligned}$$

$$= C_\Phi^N C_\Lambda^\Phi C_I^\Lambda R^I \ . \tag{18-14}$$

Since

$R^N = C_I^N R^I$

we have

$$C_I^N C_\phi^N C_\Lambda^\phi C_I^\Lambda . \tag{18-15}$$

It follows using (18-8), (18-10), and (18-12):

$$C_I^N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}_\Phi^N \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}_\Lambda^\Phi \begin{pmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}_I^\Lambda$$

or

$$C_I^N = \begin{pmatrix} -\sin\phi\cos\lambda_c & -\sin\phi\sin\lambda_c & \cos\phi \\ -\sin\lambda_c & \cos\lambda_c & 0 \\ -\cos\phi\cos\lambda & -\cos\phi\sin\lambda_c & -\sin\phi \end{pmatrix}_I^N . \tag{18-16}$$

Taking the transpose of (16) (exchanging rows and columns) will give C_N^I as given in (18-7).

18.3 DETERMINATION OF INITIAL VELOCITY $P_I R_{EP}(0)$

Applying Coriolis law to R_{EP} between the I-frame and the E-frame (see Section 7.0):

$$P_I R_{EP} = P_E R_{EP} + W_{IE} \times R_{EP} . \tag{18-17}$$

Substituting (18-4) into the right-hand side of (18-17) for R_{EP},

$$P_I R_{EP} = P_E(R_{ES} + R_{SP}) + W_{IE} \times (R_{ES} + R_{SP})$$

$$= P_E R_{ES} + P_E R_{SP} + W_{IE} \times R_{ES} + W_{IE} \times R_{SP} . \tag{18-18}$$

Referring to the second term on the right-hand side of (18-18), applying the Coriolis law between the E-frame and the S-frame,

$$P_E R_{SP} = P_S R_{SP} + W_{ES} \times R_{SP}$$

$$= W_{ES} \times R_{SP} \tag{18-19}$$

because R_{SP} is fixed in the S-frame, thus making $P_S R_{SP} = 0$.

Substituting (19 18-) in (18-18) and noting that $W_{IE} + W_{ES} = W_{IS}$, we have

$$P_I R_{EP} = P_E R_{ES} + W_{ES} \times R_{SP} + W_{IE} \times R_{ES} + W_{IE} \times R_{SP}$$

$$= P_E R_{ES} + W_{IE} \times R_{ES} + W_{IS} \times R_{SP} . \tag{18-20}$$

Evaluating (18-20) at launch (t = 0) and expressing it in the I-frame because that is the frame we use for the missile guidance and navigation, we have

$$\left(P_I R_{EP}\right)_0^I = \left(P_E R_{ES}\right)_0^I + \left(W_{IE} \times R_{ES}\right)_0^I + \left(W_{IS} \times R_{SP}\right)_0^I \tag{18-21}$$

Equation (18-16) shows that the velocity of the missile at launch relative to the center of the earth in the inertial space may be determined in terms of the values of the three terms on the right-hand side of (18-21).

We deal with them one by one.

First, $\left(P_E R_{ES}\right)_0^I$ is the initial velocity of submarine or aircraft relative to the earth-fixed frame with components expressed in the I-frame. The term $\left(P_E R_{ES}\right)_0^N$ is the initial value (at t = 0) of the submarine's velocity relative to the earth-fixed frame expressed in the N-frame. It is an output from INS and is transformed from the N-frame to the I-frame by

$$\left(P_E R_{ES}\right)_0^I = C_N^I(0)\left(P_E R_{ES}\right)_0^N$$

where $C_N^I(0)$ is the value of C_N^I at t = 0.

Next, for $\left(W_{IE} \times R_{ES}\right)_0^I$, we determine $W_{IE}^E \times R_{ES}^E$ in E-frame and transform it to the I-frame. In the E-frame 18-

$$W_{IE}^E = \begin{pmatrix} 0 \\ 0 \\ W \end{pmatrix} \text{ in which } |W_{ie}| = W$$

where W is the magnitude of the earth-rate.

Thus,

$$\left(W_{IE} \times R^{E}_{ES}\right)_0 = \begin{vmatrix} i & j & k \\ 0 & 0 & w \\ X_0 & Y_0 & Z_0 \end{vmatrix}_{E\text{-frame}} \tag{18-24}$$

$= -iwy_0 + jwx_0$
in which

$$R^{E}_{ES} = \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$$

is available at an intermediate point of the computation from INS.

Then $\left(W_{IE} \times R_{ES}\right)^{I}_{0}$ is obtained simply by

$$\left(W_{IE} \times R_{ES}\right)^{I}_{0} = C^{I}_{E}(0)\left(W^{E}_{IE} \times R^{E}_{ES}\right)^{E}_{0} \tag{18-25}$$

where $C^{I}_{E}(0)$ is the coordinate transformation matrix from the E-frame to I-frame at launch (at $t = 0$).

In our case both the I-frame and the E-frame are polar-geocentric, thus sharing the Z (polar) axis, Thus C^{I}_{E} is

$$C^{I}_{E}(\tau) = \begin{pmatrix} \cos w\tau & -\sin w\tau & 0 \\ \sin w\tau & \cos w\tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{18-26}$$

derived by rotating the E-frame about the polar axis relative to the I-frame by an angle of $W\tau$, where *w* is the magnitude of the earth rate and τ is the time elapsed since the two frames coincided (i.e., X_E axis with X_I axis and Y_E axis with Y_E axis while the Z_E axis always remains identical with the Z_I axis).

Finally to evaluate the last term on the R.H.S of (18-21), $\left(W_{IS} \times R_{Sp}\right)^{I}_{0}$, we determine it first in S–frame and then transform it to the I-frame via the N-frame. That is,

$$\left(W_{IS} \times R_{Sp}\right)^{I}_{0} = C^{I}_{N}(0)C^{N}_{S}(0)\left(W^{S}_{IS} \times R^{S}_{Sp}\right)^{S}_{0} \tag{18-27}$$

In (18-27), R^{S}_{Sp} is simply the missile (its guidance system's) location in the ship expressed in S–frame. W^{S}_{IS} is determined by the guidance system by differentiating with respect to time its gimbal angles, which are the measure of the angular deviations between the guidance system's inertially nonrotating frame and the missile structure at launch time, which is fixed to the ship. $C^{N}_{S}(0)$ is the coordinate transformation matrix from the S-frame to the N-frame at launch, which is determined based on the SINs gimbal angles (which is the measure of angular deviation between S-frame and N-frame). $C^{I}_{N}(0)$ is determined from the latitude and celestial longitude as given in (18-7).

19.0 CROSS PRODUCT STEERING (FOR ROCKET VEHICLE GUIDANCE)

Let us denote the missile's current velocity by V_M, the required velocity (such as correlated velocity) by V_R, and the velocity to be gained (or to go) by V_G, so that V_M becomes V_R as V_G goes to zero.
Then,

$$V_G = V_R - V_M \,. \tag{19-1}$$

Thus, if we drive V_G to zero with thrust, then V_M approaches V_R. It is intuitively obvious that V_G will decrease most effectively if thrust is directed parallel to the V_G direction.

In fact, it turns out that, in all current high-acceleration rocket missiles, if we steer (rotate) the missile's thrust vector with the same angular velocity w as the V_G vector, then V_G will be driven to zero, thus guaranteeing that the missile will achieve the required velocity V_R. To obtain the mathematical expression for this w, denote the unit vector along V_G by $\underline{u}$,

or

$$\underline{u} = \frac{\underline{V}_G}{V_G} \tag{19-2}$$

By differenting (19-3) WRT time:

$$\dot{\underline{u}} = \frac{V_G \dot{\underline{V}}_G - \underline{V}_G \dot{V}_G}{V_G^2} = \frac{V_G \dot{\underline{V}}_G}{V_G^2} - \frac{\underline{V}_G \dot{V}_G}{V_G^2} \tag{19-4}$$

Since $\underline{u}$ is a unit vector, its magnitude remains to be one, while its direction may change. Thus, by denoting the angular velocity of the $\underline{u}$-vector with respect to the inertial space by w, we have, as derived in the section on theorem of Coriolis,

$$\dot{\underline{u}} = \underline{w} \times \underline{u} \tag{19-5}$$

where w is the angular velocity vector of $\underline{u}$ with its rotation axis perpendicular to $\underline{u}$.

It follows from (19-5):

$$\underline{u} \times \dot{\underline{u}} = \underline{u} \times (\underline{w} \times \underline{u}) \tag{19-6}$$

Using the vector identity

$$\underline{V}_1 \times (\underline{V}_2 \times \underline{V}_3) = (\underline{V}_1 \cdot \underline{V}_3)\underline{V}_2 - (\underline{V}_1 \cdot \underline{V}_2)\underline{V}_3 \tag{19-7}$$

we have from (19-6):

$$\underline{u} \times \dot{\underline{u}} = (\underline{u} \cdot \underline{u})\underline{w} - (\underline{u} \cdot \underline{w})\underline{u} = \underline{w} \tag{19-8}$$

because $u \cdot u = 1$ and $u \cdot w = 0$ since w is perpendicular to u as mentioned above.

Substituting (19-3) and (19-4) into (19-8):

$$\underline{w} = \underline{u} \times \dot{\underline{u}}$$

$$= \frac{\underline{V}_G}{V_G} \times \left(\frac{V_G \dot{\underline{V}}_G}{V_G^2} - \frac{\underline{V}_G \dot{V}_G}{V_G^2} \right) \tag{19-9}$$

$$= \frac{\underline{V}_G}{V_G} \times \frac{V_G \dot{\underline{V}}_G}{V_G^2} - \frac{\underline{V}_G}{V_G} \times \frac{\underline{V}_G \cdot \dot{V}_G}{V_G^2} \tag{19-10}$$

The second term of (19-10) is zero because $\underline{V}_G \times \underline{V}_G = 0$. It follows:

$$\underline{w} = \frac{\underline{V}_G \times \dot{\underline{V}}_G}{V_G^2} \tag{19-11}$$

In (19-5), (19-8), and (19-11), we conclude that w is proportional to the angular velocity of the unit vector $\underline{V}_G$. Equation (19-11) shows that $\underline{w}$ may be determined from $\underline{V}_G$ and $\dot{\underline{V}}_G$. And, if the rocket's thrust vector is steered with the angular velocity of $\underline{w}$ as determined by (19-11), V_G will be most effectively nulled assuring V_M will become V_R.

In practice, (19-11) is used in the form of

$$\underline{w} = K_1 \left(\underline{V}_G \times \dot{\underline{V}}_G \right) \tag{19-12}$$

where K_1 is adjusted for good performance and may or may not be a function of $\underline{V}_G$.

Denoting the missile's roll axis by $i_B x$, pitch axis by $j_B y$ and yaw axis by $k_B z$, in the missile's body-fixed frame, we have

$$\underline{V}_G \times \underline{\dot{V}}_G = \begin{vmatrix} i_B & j_B & k_B \\ V_{GX} & V_{GY} & V_{GZ} \\ \dot{V}_{GX} & \dot{V}_{GY} & \dot{V}_{GZ} \end{vmatrix}$$

$$\begin{aligned} &= i_B\left(V_{GY}\dot{V}_{GZ} - V_{GZ}\dot{V}_{GY}\right) \\ &+ j_B\left(V_{GZ}\dot{V}_{GX} - V_{GX}\dot{V}_{GZ}\right) \\ &+ k_B\left(V_{GX}\dot{V}_{GY} - V_{GY}\dot{V}_{GX}\right). \end{aligned} \tag{19-13}$$

Thus, the commanded angular velocity in pitch and yaw directions are:

$$w_{pitch} = -K_1\left(V_{GX}\dot{V}_{GZ} - V_{GZ}\dot{V}_{GX}\right) \quad \text{along } j_B\text{-axis} \tag{19-14}$$

$$w_{yaw} = K_1\left(V_{GX}\dot{V}_{GY} - V_{GY}\dot{V}_{GX}\right) \quad \text{along } k_B\text{-axis} \tag{19-15}$$

w_{roll} does not effect the direction of thrust, and, therefore, is not used.

In the LRBM, w is determined by

$$\underline{W} = K_2(\underline{A}_T \times \underline{V}_G) \tag{19-16}$$

instead of (19-12), where A_T is the nongravitational specific force f_{NG}, which is equal to the thrust in a vacuum. To see that (19-16) is a modification of (19-12), we start with the differentiation of (19-1):

$$\underline{\dot{V}}_R = \underline{\dot{V}}_M + \underline{\dot{V}}_G \tag{19-17}$$

Since

$$\underline{\dot{V}}_M = g + f_{NG} = g + A_T \tag{19-18}$$

and noting that V_R is a very slowly varying function and thus $\underline{\dot{V}}_R$ may be approximated by zero, and also noting that $|\underline{A}_T| >> |\underline{g}|$ near the deployment release point we have from (19-17) and (19–18):

$$\underline{\dot{V}}_G = -\dot{V}_M = -g - A_T \cong -A_T\,. \tag{19-19}$$

Substituting (19-19) into (19-12):

$$W = K(\underline{V}_G \times (-\underline{A}_T)) = K(t)(\underline{A}_T \times \underline{V}_G) \tag{19-20}$$

which is the same as (19-12), with K now being a function of time for optimum performance.

20.0 DERIVATION OF WEIGHTING MATRIX W FOR STELLAR SIGHTING

20.1 INTRODUCTION

The position and velocity of the missile as determined by the onboard guidance system during powered flight have inherent errors caused by the errors, which exists in the accelerometers and gyros, as well as those caused by the launch-time initial conditions associated with navigation and fire control.

By means of a stellar sighting of a predesignated star before to the initiation of the deployment of the reentry bodies, the guidance system measures the change in elevation ΔE and change in bearing ΔB of the star relative to the guidance system platform axes.

The guidance system uses these measurements to make the best estimate, in the least square sense, of the errors in position, velocity, and platform orientation by using a weighting matrix denoted as the W matrix. The W matrix is predetermined before launch, based on the error statistics of the accelerometers and gyros as well as those of the initial conditions related to navigation and the fire control.

20.2 FORMULATION

Define a state vector x, which is to be estimated at the time t_s of the stellar sighting by:

$$X = \begin{pmatrix} \Delta p_1 \\ \Delta p_2 \\ \Delta p_3 \\ \Delta \dot{p}_1 \\ \Delta \dot{p}_2 \\ \Delta \dot{p}_3 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \qquad (20\text{-}1)$$

where Δp_1, Δp_2, and Δp_3 are the guidance system's position errors, $\Delta \dot{p}_1$, $\Delta \dot{p}_2$, and $\Delta \dot{p}_3$ are velocity errors and θ_1, θ_2, and θ_3 are platform tilt errors, along the X, Y, and Z axes, denoted by subscripts 1, 2, and 3 for convenience.

We denote a constant error vector with some 65 elements by e. Each element of the e vector is considered as a random constant for a given missile flight, but varies from flight-to-flight randomly with known standard deviations and mean values.

We relate the state X at the time of sighting to e by means of the M-matrix, that is X=Me. Note that X being a (9 by 1) column vector and e being (65 by 1) column vector, the size of the M-matrix must be (9 by 65) to be conformable.

In the stellar sighting, the onboard instrument measures the elevation error ΔE, and the bearing error ΔB (of the designated star) relative to the guidance system's platform axes. We represent this by the 2x1 measurement vector Z by:

$$Z = \begin{pmatrix} \Delta E \\ \Delta B \end{pmatrix} \tag{20-2}$$

We relate Z to the state vector X by:

$$Z = HX + v \tag{20-3}$$

where v is a 2x1 measurement noise vector, and H is called the measurement matrix. Since Z is a 2x1 matrix and X is a 9x1 matrix, H must be a 2x9 matrix to be conformable.

Our objective is to find the weighting matrix W such that the estimate $\hat{X}$ of X based on the measurement Z is optimal (in a sense to be defined later). That is, we have to determine the weighting matrix W such that $\hat{X}$ in

$$\hat{X} = WZ \tag{20-4}$$

is optimal.

It follows, substituting (20-4) into (20-5):

$$\hat{X} = W(HX + v) \tag{20-5}$$

20.3 DETERMINATION OF W MATRIX

The state vector X represents the true value of the errors in the guidance system's determination of position, velocity, and platform alignment. We seek the best (optimal) estimate $\hat{X}$ of these guidance system errors. Denoting by $\tilde{X}$ the error in the process of obtaining the estimate $\hat{X}$ of the system error, we may write

$$\hat{X} = X + \tilde{X} \tag{20-6}$$

It follows using (2) and (5) in (6) noting X = Me:

$$\tilde{X} = \hat{X} - X$$
$$= W(HX + v) - X$$
$$= WHMe + Wv - Me$$
$$= (WH - I)\,Me + Wv \tag{20-7}$$

where I is an identitiy matrix. We denote the covariance matrix of $\tilde{X}$ by $P_{\tilde{X}}$. That is:

$$P_{\tilde{X}} \triangleq E\left(\tilde{X}\tilde{X}^T\right) \tag{20-8}$$

Where E denotes the statistical expectation and the superscript T denotes a transpose.

Substituting (20-7) into (20-8):

$$P_{\tilde{X}} = E\left(\tilde{X}\tilde{X}^T\right)$$

$$= E\left\{\left[(WH-I)Me + Wv\right]\left[(WH-I)Me + Wv\right]^T\right\}$$
$$= E\left\{\left[(WH-I)Me + Wv\right]\left[e^T M^T (WH-I)^T + v^T W^T\right]\right\} \tag{20-9}$$
$$= E\left\{\left[(WH-I)Mee^T M^T (WH-I)^T\right] + Wvv^T W^T\right\}$$

with the assumption that

$$\left.\begin{aligned} E\left(ve^T\right) = 0 \\ E\left(ev^T\right) = 0 \end{aligned}\right\} \tag{20-10}$$

This is a reasonable assumption because in reality, e and v are unlikely to be correlated. Note that the expectation E operates only on the random vectors such as v and e, but not on the constant matrices such as W, M, or (WH-I).

From (20-10):

$$P_{\tilde{X}} = E\left[(WH-I)ME\left(ee^T\right)M^T (WH-I)^T + WE\left(vv^T\right)W^T\right]$$

$$= (WH-I)MP_E M^T (WH-I)^T + WP_V W^T \tag{20-11}$$

where

$$\left.\begin{aligned} P_E \underline{\underline{\Delta}} E\left(ee^T\right) \\ P_V \underline{\underline{\Delta}} E\left(vv^T\right) \end{aligned}\right\} \tag{20-12}$$

We want to determine W such that the scalar sum of the diagonal elements of $P_{\tilde{x}}$ called trace (tr) of square matrix $P_{\tilde{x}}$ is minimized because, generally, the diagonal elements are most significant and predominant. To achieve this, we set

$$\frac{\partial}{\partial W} t_r P_{\tilde{X}} = 0 \tag{20-13}$$

and we solve the resulting equation for W.

For those who are interested in a tutorial derivation of (20-16) to (20-22), see the last paragraph of this report.

From (20-11)

$$\begin{aligned} P_{\tilde{X}} &= (WH - I)\left(MP_E M^T H^T W^T - MP_E M^T\right) + WP_V W_T \\ &= W\left(HMP_E M^T H^T\right)W^T - MP_E M^T H^T W^T \\ &\quad -WHMP_E M^T + MP_E M^T - WP_V W^T \end{aligned} \tag{20-14}$$

Now, since P_E is a covariance matrix, $P_E = P_E^T$ and we have:

$$\begin{aligned} \left(HMP_E M^T H^T\right)^T &= \left(M^T H^T\right)^T \left(HMP_E\right)^T \\ &= \left(H^T\right)^T \left(M^T\right)^T P_E^T \left(HM^T\right) \\ &= HMP_E M^T H^T \end{aligned} \tag{20-15}$$

which shows it is a symmetric matrix.

According to a matrix formula , denoting trace by tr:

$$\begin{aligned} \frac{\partial}{\partial W} \operatorname{tr}\left(WBW^T\right) &= W\left(B + B^T\right) \\ &= 2WB \ \text{ if } \ B = B^T \end{aligned} \tag{20-16}$$

Thus:

$$\frac{\partial}{\partial W} \mathrm{tr}\, W\left(HMP_EM^TH^T\right)W^T = 2W\left(HMP_EM^TH^T\right) \tag{20-17}$$

and

$$\frac{\partial}{\partial W} \mathrm{tr}\, WP_VW = 2WP_V \tag{20-18}$$

since $P_V = P_V^T$.

Now, by formula:

$$\frac{\partial}{\partial W} \mathrm{tr}\, AW^T = A. \tag{20-19}$$

Thus

$$\frac{\partial}{\partial W} \mathrm{tr}\, \left(MP_EM^TH^T\right)W^T = MP_EM^TH^T \tag{20-20}$$

and also, by formula:

$$\frac{\partial}{\partial W} \mathrm{tr} WC = C^T \tag{20-21}$$

Thus:

$$\begin{aligned} \frac{\partial}{\partial W} \mathrm{tr}\, W\left(HMP_EM^T\right) &= \left(HMP_EM^T\right)^T \\ &= \left(M^T\right)^T\left(HMP_E\right)^T \\ &= M(P_E)^T(HM)^T \\ &= MP_EM^TH^T \end{aligned} \tag{20-22}$$

since $P_E = P_E^T$.

Note that, since MP_EM^T is not a function of W,

$$\frac{\partial}{\partial W} \mathrm{tr}\left(MP_EM^T\right) = 0 . \tag{20-23}$$

Thus, from (20-14) using the above equations, and setting the result equal to zero:

$$\frac{\partial}{\partial W}\operatorname{tr}P_{\tilde{X}} = 2W_0HMP_EM^TH^T - 2MP_EM^TH^T + 2W_0P_V = 0 \qquad (20\text{-}24)$$

with W replaced by W_o, the optimal value of W.

Solving (20-24) for W_o:

$$W_0\left(HMP_EM^TH^T + P_V\right) = MP_EM^TH^T \qquad (20\text{-}25)$$

or

$$W_0 = MP_EM^TH^T\left(HMP_EM^TH^T + P_V\right)^{-1} \qquad (20\text{-}26)$$

or

$$W_0 = MP_E(HM)^T\left[\left((HM)P_E(HM)^T + P_V\right)^{-1}\right] \qquad (20\text{-}27)$$

if the inverse exists. This is the W-matrix we are seeking. Further analysis shows that W_0 given in (20-26) is indeed a minimum as required.

The optimal estimate $\hat{X}$ is obtained by (20-5):

$$\hat{X} = W_0Z$$

with W_o given by (20-26)

For the tutorial, heuristic derivations of the matrix formula used in this section, demonstrated by means of 2x2 matrices for satisfaction (not as a rigorous derivation), the readers may refer to document NAVSWC MP 91-03 "Derivation of Discrete Kalman Filter Equations using Heuristic, Tutorial Approach" (Revision A) by Kee Soon Chun, NSWC, Jan 1991. Anyone interested in having a copy may contact the author at NSWCDD (Code K44).

21.0 SCHULER PERIOD AND SCHULER TUNING

Consider a spherical, nonrotating earth. If one stands still on the earth and holds a plumb-bob, it will point to the center of the earth in the vertical direction. If one accelerates, we know by experience, the line of the plumb-bob will lag behind (from the stationary direction) because of the inertial reaction. Thus, we can see that the plane perpendicular to the plumb-bob line cannot serve as the true horizontal plane because when accelerating, the plumb-bob line deviates from the true vertical. That is, the reference frame constructed based on the plumb-bob is not accurate because the frame tilts under acceleration.

Question : How can we make the plumb-bob follow the vertical even under acceleration?

It is intuitively obvious that, if we can make the angular rotation of the plumb-bob line relative to its pivot point equal to the angular rotation of the earth's radius vector from the center of the earth to the pivot point, the problem is solved.

Now, consider a hypothetical simple pendulum with the length equal to the radius of the earth and the mass m located at the center of the earth. It is obvious that, no matter how the pivot accelerates, the line of the pendulum always points to the center of the earth along the line of vertical.

From elementary physics, we know the period T of a simple pendulum with length L is:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

For the pendulum with L=R, the radius of the earth:

$$T = 2\pi\sqrt{\frac{R}{g}} \cong 84\,\text{min}$$

In his classical paper (published in 1923) on the research of the gyrocompass (North-seeking device), a German scientist, Max Schuler suggested that if we could build a servo-mechanical control device (platform), which has a natural period of oscillation of 84 min, the device will track the earth's vertical under acceleration without oscillation, if it was initially aligned with the earth's vertical.

For this reason, the 84-min period is referred to as the Schuler period, and the natural frequency corresponding to the Schuler period is called the Schuler frequency. That is,

$$W_n = 2\pi\frac{1}{T} = 2\pi\frac{1}{2\pi}\sqrt{\frac{g}{R}}$$

or

$W_n = \sqrt{\frac{g}{R}}$ is the Schular frequency.

A physical device (such as an inertial platform) that has this arrangement is said to be "Schuler tuned."

If the platform is Schuler tuned this way, the angular velocity of the platform's vertical axis is equal to the angular velocity of the true vertical, if it is intially aligned. If it is not initially aligned, the platform's vertical axis will oscillate about the true vertical with a natural frequency of an 84-min. period and with an amplitude of oscillation equal to the initial angle of misalignment.

22.0 CORRELATED VELOCITY

22.1 INTRODUCTION

The correlated velocity is the required velocity of a reentry vehicle (RV), which will hit the target on the rotating earth by free-falling in a specified time of flight (t_f). It is the required velocity correlated with the target displacement in the inertial space caused by the rotation of earth during the specified TOF of the RV.

The specification of t_f is necessary because the target is located on the rotating earth and thus moves to a new location in the inertial space during t_f, and because we want the RV to hit the same target at the new location in the inertial space.

Another way of looking at the problem is how much speed and in what direction the RV should have so that, by free-falling along a elliptic trajectory, it will intercept the target, which is moving along a circular orbit in the inertial space, while still fixed on the earth.

Referring to Figure 22-1, the problem of finding the correlated velocity may be stated as follows:

Given: $r(0) = r_0$ at $t=0$ of RV,
$r_T(0)$ of target at $t=0$ fixed on the rotating earth
t_f time of flight of free-falling RV released at $t = 0$ until it hit the target

Find: V_c (Correlated Velocity magnitude) and γ_c (flight path angle) such that
$\underline{r}(t_f) = \underline{r}_T$ at $t = t_f$
that is, such that the position of RV coincide with the position of target on the rotating earth at $t = t_f$ or, RV intercepts the target

This problem is known as Lambert's problem or the Lambert Guidance problem.

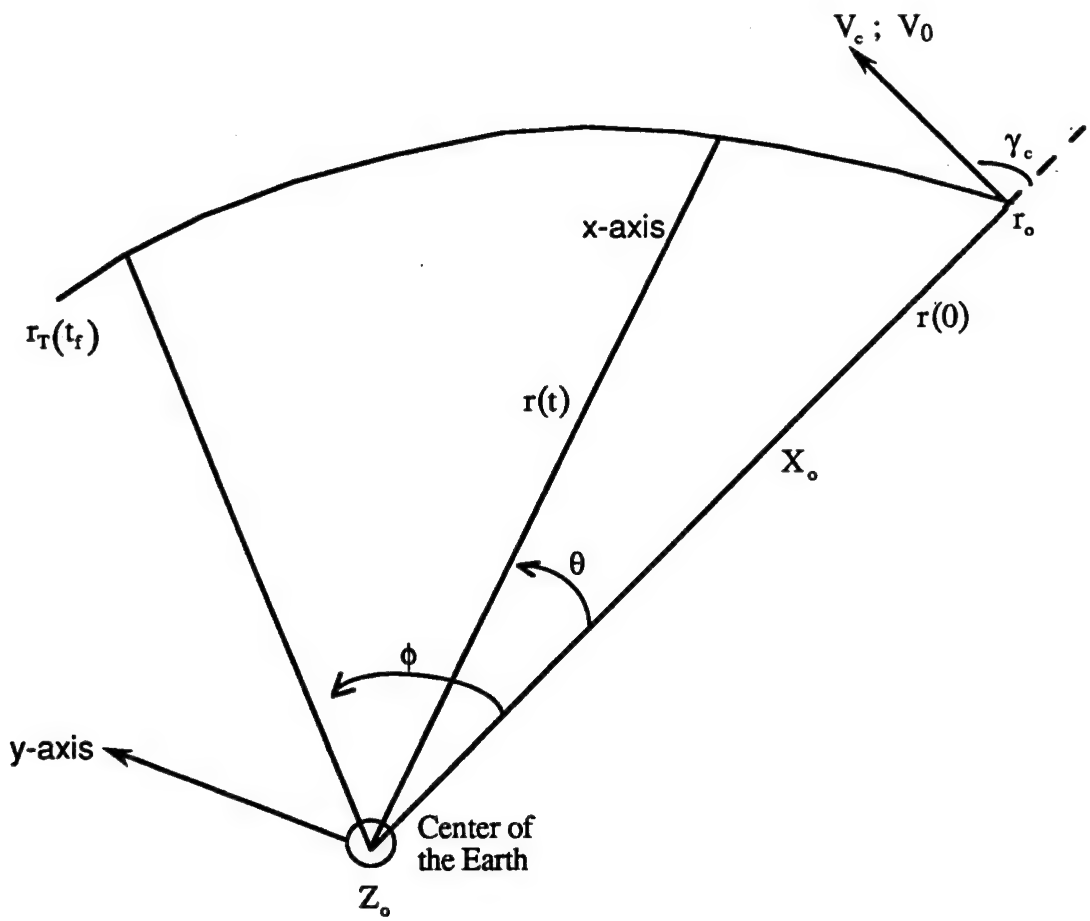

FIGURE 22-1. CORRELATED VELOCITY

The derivation of equation to determine the correlated velocity, in the approach adopted in this report, in terms of r_0 and $r_T(t_f)$ is based on two key equations.

The first equation is the expression for the conservation of angular momentum along the line normal to the plane containing the RV's trajectory. The second equation is the equation of motion for the RV derivable from Newton's second law of motion along the radial direction of the RV from the center of the earth.

22.2 CONSERVATION OF ANGULAR MOMENTUM (refer to Section 1.0 for notations)

Applying Newton's second law to the RV mass, m

$$mP_I^2\underline{r} = -\frac{GMm}{r^2}\frac{\underline{r}}{r} \tag{22-1}$$

where $P_I(\cdot)$ represents $\frac{d}{dt}(\cdot)$ in inertial space and M is the mass of the earth that is assumed to be concentrated at its center. Using $P_I\underline{r}=\dot{\underline{r}}$, $P_I^2 r=\ddot{\underline{r}}$ and $\mu \equiv GM$ in (22-1):

$$P_I^2 r=\ddot{\underline{r}}=-\frac{\mu}{r^3}\underline{r} \tag{22-2}$$

multiplying both sides of (22-2) by $\underline{r}\times$ (vector cross-product):

$$\underline{r}\times\ddot{\underline{r}} =-\frac{\mu}{r^3}\underline{r}\times\underline{r} \tag{22-3}$$

Noting that $\underline{r}\times\underline{r} = 0$, we have from (22-3):

$$\underline{r}\times\ddot{\underline{r}}=0 \tag{22-4}$$

Since, by simple differentiation

$$\frac{d}{dt}\underline{r}\times\dot{\underline{r}}=\dot{\underline{r}}\times\dot{\underline{r}}+\underline{r}\times\ddot{\underline{r}} \tag{22-5}$$

and since $\dot{\underline{r}}\times\dot{\underline{r}}=\underline{0}$, we have using (22-4):

$$\frac{d}{dt}(\underline{r}\times\dot{\underline{r}})=0 \tag{22-6}$$

or

$$\underline{r}\times\dot{\underline{r}}=\underline{h}=\text{constant vector.} \tag{22-7}$$

This means for anytime t,

$$\begin{aligned}\underline{r}(0)\times\dot{\underline{r}}(0)&=\underline{r}(t)\times\dot{\underline{r}}(t)\\ &=\underline{h}\end{aligned} \tag{22-8}$$

To proceed further, we define a new rotating frame of reference that we call the T-frame. Referring to Figure 22-1, the x-axis is along the r(t) direction and the y-axis is along the line perpendicular (in counter-clockwise direction) to r(t) on the trajectory plane. The z-axis is normal to the x-y plane (trajectory plane) with the origin at the center of the earth sharing the origin with the earth-centered I-frame. The T-frame rotates about the z-axis with the angular velocity $W_{IT}=\dot{\underline{\theta}}$ of the T-frame relative to the earth centered I-frame.

Since the $\underline{h}$ vector is by definition of cross product, perpendicular to both $\underline{r}$ and $\dot{\underline{r}}$, it is perpendicular to the plane formed by $\underline{r}$ and $\dot{\underline{r}}$. Since $\underline{h}$ is constant, the plane containing $\underline{r}$ and $\dot{\underline{r}}$ is constant as well for all t, which means that the trajectory of RV remain on a fixed plane.

Applying the law of Coriolis to r(t) between the I-frame and the T-frame (refer to the section on the law of Coriolis for derivation):

$$P_I r(t) = P_T r(t) + W_{TI} \times r(t) \tag{22-9}$$

where

$P_I(\cdot) = \dfrac{d_I}{dt}(\cdot)$ — time rate of change of vector (·) as observed in the I-frame.

$P_T(\cdot) = \dfrac{d_T}{dt}(\cdot)$ — time rate of change of vector (·) as observed in the T-frame.

In the T-frame, according to our definition: (Note: superscript T should not be confused with transpose.)

$$\underline{r}^T(t) = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}^T \; ; \; P_T r(t) = \begin{pmatrix} \dot{r} \\ 0 \\ 0 \end{pmatrix}^T \tag{22-10}$$

$$W_{TI}^T = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}^T \tag{22-11}$$

so that

$$\underline{W}_{TI} \times \underline{r}(t) = \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta} \\ r & 0 & 0 \end{vmatrix}_T = \begin{pmatrix} 0 \\ r\dot{\theta} \\ 0 \end{pmatrix}^T \text{ expressed in T–frame .} \tag{22-12}$$

Substituting (22-10) and (22-12) into (22-9) :

$$[P_I r]^T = \dot{\underline{r}}^T = \begin{pmatrix} \dot{r} \\ 0 \\ 0 \end{pmatrix}^T + \begin{pmatrix} 0 \\ r\dot{\theta} \\ 0 \end{pmatrix}^T = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ 0 \end{pmatrix}^T \tag{22-13}$$

with components expressed in the trajectory frame (T-frame).

It follows, expressed in the T-frame:

$$h^T = \underline{r}^T \times \dot{\underline{r}}^T = \begin{vmatrix} i & j & k \\ r & 0 & 0 \\ \dot{r} & r\dot{\theta} & 0 \end{vmatrix}_T = \begin{pmatrix} 0 \\ 0 \\ r^2\dot{\theta} \end{pmatrix}^T \tag{22-14}$$

According to (22-14), the angular momentum viewed from the trajectory plane is pointed along the T-frame's z-axis as it should be and is equal to

$$h = r^2\dot{\theta} = r^2 \frac{d\theta}{dt} \ . \tag{22-15}$$

From (22-15), we get

$$\boxed{dt = \frac{r^2}{h} d\theta} \tag{22-16}$$

(22-16) is the first of two key equations necessary to derive the equation to determine the correlated velocity in terms of known quantities.

<u>EQUATION OF MOTION</u>—along the x-axis (radial direction) of the trajectory (T) frame.

Operating on both sides of (22-9) by P_I :

$$P_I(P_I r) = P_I(P_T r) + P_I(W_{IT} \times r). \tag{22-17}$$

Referring to the right-hand side of (22-17), treating $P_T r$ and $W_{IT} \times r$ as vectors and applying Coriolis law:

$$\begin{aligned} P_I(P_T r) &= P_T(P_T r) + W_{IT} \times (P_T r) \\ &= P_T^2 r + W_{IT} \times (P_T r) \end{aligned} \qquad (22\text{-}18)$$

and

$$\begin{aligned} P_I(W_{IT} \times r) &= P_T(W_{IT} \times r) + W_{IT} \times (W_{IT} \times r) \\ &= (P_T W_{IT}) \times r + W_{IT} \times (P_T r) + W_{IT} \times (W_{IT} \times r). \end{aligned} \qquad (22\text{-}19)$$

Substituting (22-18) and (22-19) into (22-17):

$$P_I^2 r = P_T^2 r + (P_T W_{IT}) \times r + 2W_{IT} \times (P_T r) + W_{IT} \times (W_{IT} \times r). \qquad (22\text{-}20)$$

Noting that $\dot{\theta}$ is along the z-axis and thus $\underline{W}_{IT} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix}^T$, and $\underline{r} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}^T$, we have

$$P_T^2 r = P_T(P_T r) = P_T \begin{pmatrix} \dot{r} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \ddot{r} \\ 0 \\ 0 \end{pmatrix}^T \qquad (22\text{-}21)$$

$$P_T W_{IT} = P_T \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta} \end{pmatrix}^T \qquad (22\text{-}22)$$

$$(P_T W_{IT}) \times r = \begin{vmatrix} i & j & k \\ 0 & 0 & \ddot{\theta} \\ r & 0 & 0 \end{vmatrix}_T \rightarrow \begin{pmatrix} 0 \\ r\ddot{\theta} \\ 0 \end{pmatrix}^T \qquad (22\text{-}23)$$

$$W_{IT} \times (P_T r) = \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta} \\ \dot{r} & 0 & 0 \end{vmatrix}_T \rightarrow \begin{pmatrix} 0 \\ \dot{r}\dot{\theta} \\ 0 \end{pmatrix}^T \qquad (22\text{-}24)$$

and using (22-12)

$$W_{IT} \times (W_{IT} \times r) = \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta} \\ 0 & r\dot{\theta} & 0 \end{vmatrix}_T \rightarrow \begin{pmatrix} -r\dot{\theta}^2 \\ 0 \\ 0 \end{pmatrix}^T . \tag{22-25}$$

Substituting (22-21), (22-23), (22-24), and (22-25) into (22-20):

$$\left(P_I^2 r\right)^T = \begin{pmatrix} \ddot{r} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ r\ddot{\theta} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2\dot{r}\dot{\theta} \\ 0 \end{pmatrix} + \begin{pmatrix} -r\dot{\theta}^2 \\ 0 \\ 0 \end{pmatrix} \tag{22-26}$$

with components expressed in the T-frame. It follows that the x-component (in the radial direction of (22-26) is:

$$\left(P_I^2 r\right)_x^T = \ddot{r} - r\dot{\theta}^2 \tag{22-27}$$

and the y-component is

$$\left(P_I^2 r\right)_y^T = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{22-28}$$

Since the gravitational force is along the radial or x-axis:

$$\frac{\mu}{r^3}\underline{r}^T = \frac{\mu}{r^2}\frac{\underline{r}^T}{r} = \begin{pmatrix} -\frac{\mu}{r^2} \\ 0 \\ 0 \end{pmatrix} \tag{22-29}$$

where $\frac{\underline{r}}{r}$ is a unit vector along $\underline{r}$ direction.

From (22-2), (22-27), and (22-29):

$$\boxed{\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}} \tag{22-30}$$

in the radial direction or along the x-axis.

And, from (22-2), (22-28), and (22-29):

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{22-31}$$

in the direction perpendicular to the radial direction, or along the y-axis direction.

Equation (22-30) is the equation of motion of the RV along the radial direction, the second key equation we are looking for.

Referring to (22-31):

$$\begin{aligned} r\ddot{\theta} + 2\dot{r}\dot{\theta} &= \frac{1}{r}\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right) \\ &= \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0. \end{aligned} \tag{22-32}$$

From (22-32):

$$\frac{d}{dt}r^2\dot{\theta} = 0 \tag{22-33}$$

Integrating, we have $r^2\dot{\theta} = \text{constant}$. Denoting this constant by h, we have

$$r^2\dot{\theta} = h \tag{22-34}$$

which is identical with (22-15),thus reaffirming the conservation of angular momentum.

Defining a new variable u by

$$u \triangleq \frac{1}{r} \quad \text{or} \quad r = \frac{1}{u} \tag{22-35}$$

and substituting in (22-34):

$$\dot{\theta} = hu^2 \tag{22-36}$$

Now

$$\begin{aligned} \dot{r} &= \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \\ &= \dot{\theta}\frac{dr}{d\theta} = \dot{\theta}\frac{d}{d\theta}\left(\frac{1}{u}\right) \end{aligned}$$

Using (22-36) in the above equation :

$$\dot{r} = -h\frac{du}{d\theta} \tag{22-37}$$

Also,

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{r}}{d\theta}$$

Using (22-36) and (22-37) in the above equation:

$$\ddot{r} = -h^2 u^2 \frac{d^2 u}{d\theta^2} \tag{22-38}$$

Finally, using (22-35), (22-36), and (22-38) in (22-30):

which is a linear differential equation equivalent to (22-30).

$$\boxed{\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2}} \tag{22-39}$$

22.3 CORRELATED VELOCITY

This section is a detailed version of the method of determining the correlated velocity described in Reference (7),with the gaps and details filled in by the author of this report.

There are several approaches to the solution of the problem. Although the method adopted here is not efficient in computation time, it was chosen because it is, from the author's viewpoint, the least difficult and least prone to losing the insights into what is going on. The most efficient method may be somewhat difficult in providing insight for beginners. Those readers, who are interested in further study of the subject, are referred to an excellent book by Richard H. Battin entitled " An Introduction to the Mathematics and Methods of Astrodynamics" (AIAA Education Series, 1987.), which must be counted among the great classics in scientific literature as the Editor-in-Chief of the series noted.

Recalling that (Figure 22-1)

$$\underline{h} = \underline{r}_0 \times \underline{\dot{r}} = r_0 V_0 \sin\gamma \tag{22-40}$$

and referring to the $\frac{\mu}{h^2}$ term in (22-39), we may write

$$\frac{\mu}{h^2} = \frac{\mu}{r_0^2 V_0^2 \sin^2\gamma}$$

$$= \frac{1}{\frac{r_0 V_0^2}{\mu} r_0 \sin^2\gamma}$$

$$= \frac{1}{\lambda r_0 \sin^2\gamma} \tag{22-41}$$

where

$$\lambda \triangleq \frac{r_0 V_0^2}{\mu} \tag{22-42}$$

Substituting (22-41) into (22-39) :

$$\frac{d^2 u(\theta)}{d\theta} + u(\theta) = \frac{1}{\lambda r_0 \sin^2\gamma} \ . \tag{22-43}$$

Referring to Figure 22-1, assuming $\theta = 0$ when $t = 0$ and $r(0) = r_0$, the initial conditions necessary for the solution of (22-43) are:

$$u(0) = \frac{1}{r(0)} \tag{22-44}$$

and referring to Figure 22-2,

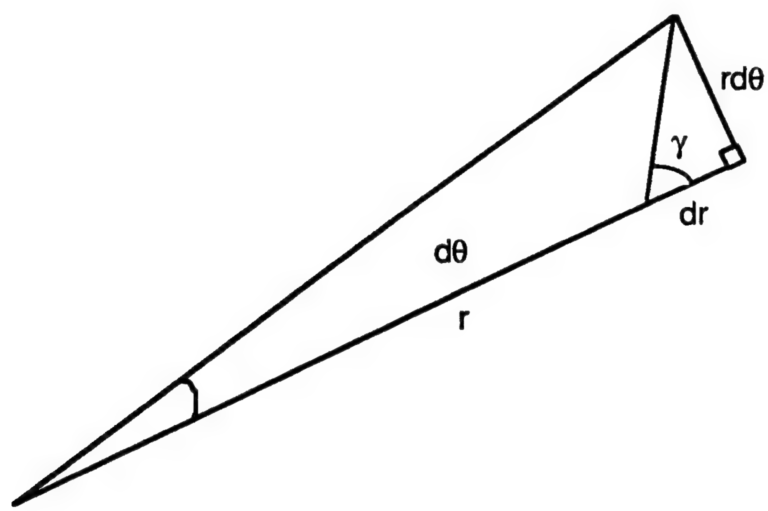

FIGURE 22-2 DEPICTION OF $\cot\gamma = \frac{dr}{rd\theta}$

$$\begin{aligned} \frac{d}{d\theta} u(\theta) &= \frac{d}{d\theta} \frac{1}{r(\theta)} \\ &= -\frac{1}{r^2} \frac{dr}{d\theta} \\ &= -\frac{1}{r} \frac{dr}{rd\theta} \end{aligned}$$

$$= -\frac{1}{r}\cot\gamma$$

Thus, since $r = r_0$ when $\theta = \theta_0$:

$$\left.\frac{du(\theta)}{d\theta}\right|_{\theta=0} = -\frac{1}{r_0}\frac{\cos\gamma}{\sin\gamma} \quad (22\text{-}45)$$

The complete solution of (22-43) is (as shown in the appendix at the end of this section):

$$u(\theta) = \frac{1}{r(\theta)} = \frac{1}{r_0}\left(\frac{1-\cos\theta}{\lambda\sin^2\gamma} + \frac{\sin(\gamma-\theta)}{\sin\gamma}\right) \quad (22\text{-}46)$$

The validy of (22-46) as the solution of (22-43) may be confirmed by differentiating $u(\theta)$ W.R.T θ,

$$\frac{du(\theta)}{d\theta} = \frac{1}{r_0}\left(\frac{\sin\theta}{\lambda\sin^2\gamma} - \frac{\cos(\gamma-\theta)}{\sin\gamma}\right) \quad (22\text{-}47)$$

and

$$\frac{d^2u(\theta)}{d\theta^2} = \frac{1}{r_0}\left(\frac{\cos\theta}{\lambda\sin^2\gamma} - \frac{\sin(\gamma-\theta)}{\sin\gamma}\right) \quad (22\text{-}48)$$

and substituting (22-46) and (22-48) into (22-43).

When θ reaches ϕ (range angle), r becomes r_T, the target vector.

Evaluating (22-46) at this boundry condition,

$$\frac{r_0}{r_T} = \left(\frac{1-\cos\phi}{\lambda\sin^2\gamma} + \frac{\sin(\gamma-\phi)}{\sin\gamma}\right) \quad (22\text{-}49)$$

Solving (22-49) for λ and recalling the definition of (22-42) :

$$\lambda = \frac{r_0 V_0^2}{\mu} = \frac{1-\cos\phi}{\left(\frac{r_0}{r_T}\right)\sin^2\gamma + \sin(\phi-\gamma)\sin\gamma} \quad (22\text{-}50)$$

Solving (50) for V_0:

$$V_0 = \left[\left(\frac{\mu}{r_0}\right) \frac{1-\cos\phi}{\left(\frac{r_0}{r_T}\right)\sin^2\gamma + \sin(\phi-\gamma)\sin\gamma} \right]^{\frac{1}{2}} \tag{22-51}$$

There are many combinations of V_0 and γ (magnitude and direction of correlated velocity) which satisfy (22-51) for given values of r_0, r_T and ϕ. To determine the unique values of V_0 and γ, we have to specify the time of (free fall) flight, which is shown next.

Referring to (22-16), integrating dt from 0 to t_f (time of flight) and $d\theta$ from 0 to ϕ (range angle), we have

$$\int_0^{t_f} dt = \frac{1}{h}\int_0^{\phi} r^2(\theta)\,d\theta \tag{22-52}$$

Substituting for $r(\theta)$ from (22-46), and for h from (22-40):

$$\boxed{t_f = \frac{r_0}{V_0\sin\gamma}\int_0^{\phi}\left[\left(\frac{1-\cos\theta}{\lambda\sin^2\gamma}\right)+\left(\frac{\sin(\gamma-\theta)}{\sin\gamma}\right)\right]^{-2} d\theta} \tag{22-53}$$

with λ given by (22-50), and V_0 given by (22-51).

Equation (22-53) shows that for the given time of flight t_f and range angle ϕ, the V_0 and γ, which satisfy (22-53) becomes the correlated velocity $V_c = V_0$ with the flight pass angle $\gamma_c = \gamma$ at $t = 0$ or at deployment.

The integrand of (22-53) may be written

$$\left[\frac{1-\cos\theta}{\lambda\sin^2\gamma}+\frac{\sin(\gamma-\theta)}{\sin\gamma}\right]^{-2} = \left[\frac{1}{\lambda\sin^2\gamma}-\frac{\cos\theta}{\lambda\sin^2\gamma}+\frac{\sin\gamma\cos\theta-\cos\gamma\sin\theta}{\sin\gamma}\right]^{-2}$$

$$= \left[\frac{1}{\lambda\sin^2\gamma}+\left(1-\frac{1}{\lambda\sin^2\gamma}\right)\cos\theta-\cot\gamma\sin\theta\right]^{-2}$$

$$= [a + b\cos\theta + c\sin\theta]^{-2} \tag{22-54}$$

where

$$a = \frac{1}{\lambda \sin^2\gamma}$$

$$b = 1 - a$$

$$c = -\cot\gamma$$

Using (22-54) in (22-53) :

$$t_f = \frac{r_0}{V_0 \sin\gamma} \int_0^{\phi} \frac{d\theta}{(a + b\cos\theta + c\sin\theta)^2} \tag{22-55}$$

The integration may be performed, resulting in a closed form solution based on the formula table.

After some algebra, the final algebraic equation for t_f may be written :

$$t_f = \frac{r_0}{V_0 \sin\gamma} \left\{ \frac{(1-\cos\phi)\cot\gamma + (1-\lambda)\sin\phi}{(2-\lambda)\left(\frac{r_0}{r_T}\right)} + \frac{2\sin\gamma}{\lambda\left(\frac{2}{\lambda}-1\right)^{3/2}} \tan^{-1}\left[\frac{((2/\lambda)-1)^{\frac{1}{2}}}{\sin\gamma\cot\left(\frac{\phi}{2}\right) - \cos\gamma} \right] \right\} \tag{22-56}$$

Referring to (22-56) and equivalently to (22-53), we note that t_f becomes infinite at $\gamma = 0$, since $\sin\gamma$ is in the denominator. However, for the current LRBM applications, γ never becomes zero as explained below.

Multiplying both sides of (22-49) by $\lambda \sin^2\gamma$, we have

$$\frac{r_0}{r_T} \lambda \sin^2\gamma = 1 - \cos\phi + \lambda \sin\gamma \sin(\gamma - \phi) \tag{22-57}$$

It follows from setting $\gamma = 0$ in (22-57) that we will have $\cos\phi = 1$ or the range angle $\phi = 0$. This means that the current position vector is $\underline{r}_0$ colinear with the target vector $\underline{r}_T$ which never happens in the existing LRBM practices. Of course, this is intuitively obvious without mathematics.

The objective of the following is to verify Equation (51) for the special case of a circular orbit (satellite injection into a circular orbit).

For a special case of a circular orbit, $|\underline{r}_0| = |\underline{r}_T|$ and γ=90°. From the range angle ϕ introductory physics, we know, for a circular orbit of r_0 that

$$\frac{\mu m}{r_0^2} = \frac{mv_0^2}{r_0} \tag{22-58}$$

It follows:

$$V_0 = \sqrt{\frac{\mu}{r_0}} \tag{22-59}$$

Evaluating (22-51) with $r_0 = r_T$ and $\gamma = 90°$ and noting that

$$\begin{aligned} \sin(\phi - \gamma) &= \sin\phi \cos 90° - \cos\phi \sin 90° \\ &= -\cos\phi \end{aligned} \tag{22-60}$$

we have

$$\begin{aligned} V_0 &= \left[\frac{\mu}{r_0}\,\frac{1-\cos\phi}{1-\cos\phi}\right]^{\frac{1}{2}} \\ &= \sqrt{\frac{\mu}{r_0}} \end{aligned} \tag{22-61}$$

Thus, verifying the validity of (22-51) for the special case of a circular orbit.

Now, for a circular orbit of period T_p, $V_0 T_p = 2\pi r_0$. That is

$$T_p = \frac{2\pi r_0}{v_0} \tag{22-62}$$

For a range angle ϕ and the corresponding time of flight t_f in a circular orbit :

$$\frac{t_f}{T_p} = \frac{\phi}{2\pi} \tag{22-63}$$

Using (22-62) in (22-63),

$$t_f = \frac{\phi r_0}{V_0} \tag{22-64}$$

Also for a circular orbit, using (22-59) in (22-42) :

$$\lambda \triangleq \frac{r_0 V_0^2}{\mu}$$

$$= \frac{r_0 \frac{\mu}{r_0}}{\mu} = 1 \tag{22-65}$$

Using $\gamma = 90°, \lambda=1$ in (22-53) and noting

$$\sin(\gamma - \theta) = \sin 90° \cos\theta - \cos 90° \sin\theta$$

$$= \cos\theta \tag{22-66}$$

we have

$$T_f = \frac{r_0}{V_0} \int_0^{\phi} \left[\frac{1}{1 - \cos\theta + \cos\theta} \right]^{-2} d\theta$$

$$= \frac{r_0}{V_0} \phi \tag{22-67}$$

which is equal to (64).

Thus verifying the validity of (53) for the special case of a circular orbit.

22.4 TRAJECTORY: A SEGMENT OF ELLIPTIC ORBIT

The general solution of (22-39) is given by

$$u(\theta) = \frac{1}{r(\theta)} = \frac{\mu}{h^2}(1 + e\cos\theta) \tag{22-68}$$

This may be verified by differentiating (22-68) :

$$\frac{du(\theta)}{d\theta} = -\frac{\mu}{h^2} e\sin\theta \tag{22-69}$$

$$\frac{d^2u(\theta)}{d\theta^2}=-\frac{\mu}{h^2}e\cos\theta \qquad (22\text{-}70)$$

and substituting (22-68) and (22-70) into (22-39), and obtaining the right-hand side of (22-39).

Rewriting (22-68) in a more conventional form:

$$r(\theta)=\frac{h^2/u}{1+e\cos\theta} \qquad (22\text{-}71)$$

In (22-71), $\frac{h^2}{\mu}=a\left(1-e^2\right)$ as shown in Appendix B. Thus, (22-71) becomes :

$$r(\theta)=\frac{a\left(1-e^2\right)}{1+e\cos\theta} \qquad (22\text{-}72)$$

which is the equation for the conic sections (see for instance, "Calculus and Analytic Geometry" by Thomas, Addison Wesley) where e represents the eccentricity of the conic sections, and a is the semi-major axis in the case of ellipses. It is well known that the conic section becomes a circle if $e = 0$, an ellipse if $0<e<1$, a parabola if $e=1$ and a hyperbola if $e>1$.

Now, solving (46) for $r(\theta)$:

$$r(\theta)=\frac{r_0\lambda\sin^2\gamma}{1+[\lambda\sin\gamma\,(\sin\gamma-\theta)-\cos\theta]} \qquad (22\text{-}73)$$

which is in the form of (22-72).

From (22-72) and (22-73), we can see that the relationships between e and λ may be determined. It turns out, after some algebra, that :

$\lambda = 1$ corresponds to $e = 0$ for circle (with $\gamma=\frac{\pi}{2}$).

$0<\lambda<2$ corresponds to $0<e<1$ for ellipse

$\lambda = 2$ corresponds to $e = 1$ for parabola

$\lambda > 2$ corresponds to $e > 1$ for hyperbola

In applications for LRBM, the combinations of r_0, v_0 and γ is such that λ is always in the range of $0<\lambda<2$, which corresponds to $0<e<1$, thus assuring the free falling ballistic trajectory is always a segment of an elliptic orbit.

22.5 COMPUTATION

The key equation for the computation of the correlated velocity is the Equation (22-56). What we want is the magnitude of the velocity V_0 and its direction γ which satisfy (22-56) for a specified time of flight t_f, with the given values of the initial release or deployment position $\underline{r}_0$, the target position $\underline{r}_T$, and the range angle ϕ between $\underline{r}_o$ and $\underline{r}_T$. Since by the defintion of the dot product

$$\underline{r}_0 \cdot \underline{r}_T = r_0 r_T \cos\phi$$

we can determine ϕ from

$$\phi = \cos^{-1} \frac{\underline{r}_0 \cdot \underline{r}_T}{r_0 \; r_T} \;.$$

Note that λ in (22-56) is given by the definition :

$$\lambda = \frac{r_0 V_0^2}{\mu} \tag{22-42}$$

This implies that V_0 appears in several places in (22-56) either explicity or implicity.

For the determination of V_0 and γ by a numerical method, we start with an initial trial guess of γ, say 45°, and get the value of V_0 corresponding to this value of γ by means of equation (22-51).

Next, substitute the values of γ and V_0 obtained above to see if the right-hand side of (22-56) matches the LHS of (22-56), which is the specified value of the time-of-flight t_f. Otherwise, we continue the process by adjusting the trial value of γ, which yields a new value of V_0, until the difference between the LHS and the RHS of (22-56) falls within the acceptable limit. The V_0 and γ, which satisfy this condition is the correlated velocity we are looking for under the specified conditions.

APPENDIX A
SOLUTION OF EQUATION (22-43)

Repeating (43):

$$\frac{d^2u(\theta)}{d\theta}+u(\theta)=\frac{1}{\lambda V_0 \sin^2\gamma}$$

Thus, with $\frac{d}{d\theta} \equiv D$,

$$\left(D^2+1\right)u=\frac{1}{\lambda r_0 \sin^2\gamma} \equiv K \qquad \text{(A-1)}$$

Let U_h denote the homogeneous solution and U_p particular solution of (A-1). Then for the homogeneous solution:

$$\left(D^2+1\right)=0$$

$$(D+i)(D-i)=0$$

$$D=i \quad D=-i$$

Thus,

$$u_h=C_1\cos\theta+C_2\sin\theta$$

For the particular solution:

Assume

$u_p = A$

then

$u_p^{\prime}=0$

$u_p^{\prime\prime}=0$

Substituting into (A-1) :

$0 + A = K$

so, the complete solution $u=u_h+u_p$ is

$$u= C_1\cos\theta+C_2\sin\theta+k \qquad \text{(A-2)}$$

Repeating (22-44)

$$u(0)=\frac{1}{r_0} \quad \text{at } \theta=0 \tag{A-3}$$

Substituting (A-2) evaluated at θ = 0 into (A-3):

$$\frac{1}{r_0}=C_1+k$$

or

$$C_1=\frac{1}{r_0}-k \tag{A-4}$$

Differentiating (A-2):

$$\frac{du}{d\theta}=-C_1\sin\theta+C_2\cos\theta. \tag{A-5}$$

Repeating (22-45):

$$\left.\frac{du}{d\theta}\right|_{\theta=0}=-\frac{1}{r_0}\frac{\cos\gamma}{\sin\gamma} \tag{A-6}$$

Substituting (A-6) into (A-5) and evaluating at θ=0:

$$C_2=-\frac{1}{r_0}\frac{\cos\gamma}{\sin\gamma} \tag{A-7}$$

Substituting (A-4) and (A-7) into (A-2) with $K=\frac{1}{\lambda r_0 \sin^2\gamma}$

$$u=\left(\frac{1}{r_0}-\frac{1}{\lambda r_0\sin^2\gamma}\right)\cos\theta-\frac{1}{r_0}\frac{\cos\gamma}{\sin\gamma}\sin\theta+\frac{1}{\lambda r_0\sin^2\gamma}$$

$$u=\frac{1}{r_0}\left[\left(1-\frac{1}{\lambda\sin^2\gamma}\right)\cos\theta-\frac{\cos\gamma}{\sin\gamma}\sin\theta+\frac{1}{\lambda\sin^2\gamma}\right]$$

$$u=\frac{1}{r_0}\left[\left(\frac{1-\cos\theta}{\lambda\sin^2\gamma}\right)+\frac{\sin\gamma\cos\theta-\cos\gamma\sin\theta}{\sin\gamma}\right]$$

$$u=\frac{1}{r_0}\left(\frac{1-\cos\theta}{\lambda\sin^2\gamma}+\frac{\sin(\gamma-\theta)}{\sin\gamma}\right)$$

which is identical with (22-46) in the main text.

APPENDIX B
EQUATION OF ELLIPSE

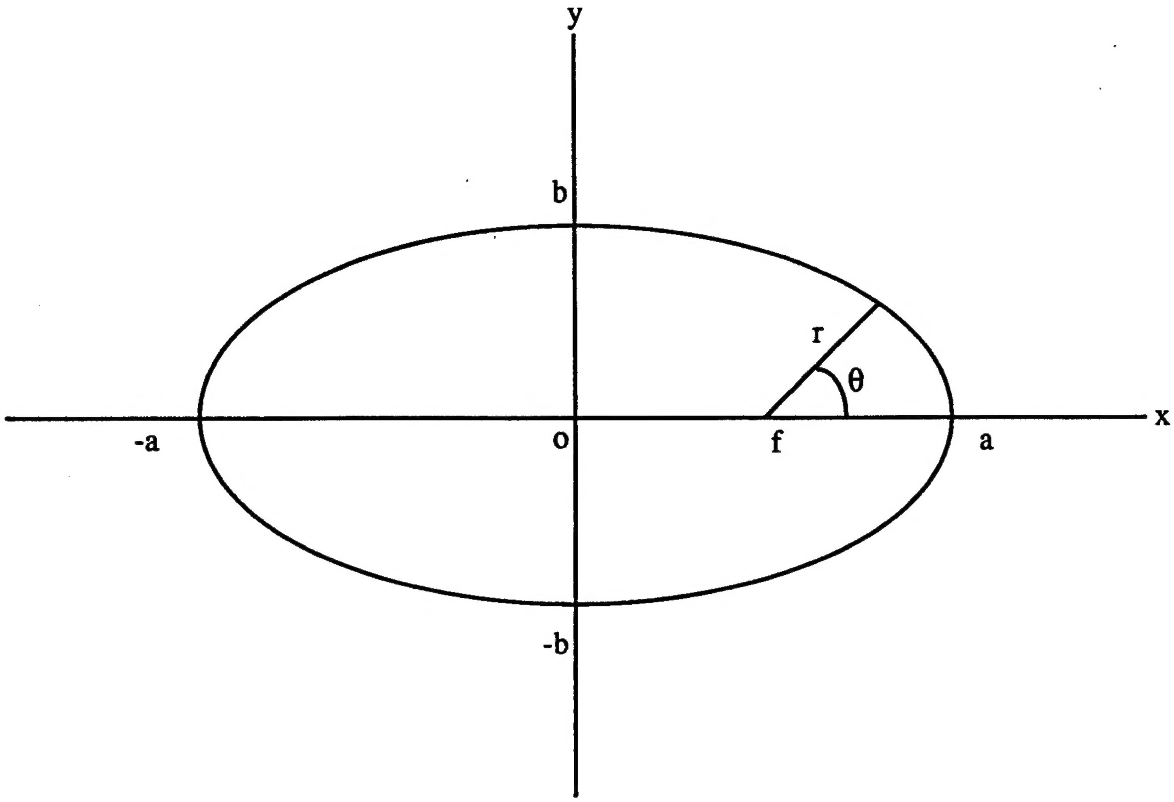

FIGURE B-1. ELLIPSE

The equation of an ellipse in rectangular coordinate is, as we all know, from analytic geometry:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{B-1}$$

The equivalent form of an ellipse in polar coordinates with $r(\theta)$ measured from one of the focal points (f) is given by :

$$r(\theta)=\frac{a\left(1-e^2\right)}{1+e\cos\theta} \tag{B-2}$$

See, for instance, Thomas: Calculus and Analytic Geometry) where the eccentricity e is the ratio of distance (of) to (oa) $\left(\text{i.e., } e=\frac{of}{oa}\right)$. The e has to be positive and less than 1 (i.e., $0 < e < 1$) to have geometrical shape of ellipse.

The semimajor axis (a) may be chosen arbitrarily as long as it is a positive real number. So, we

pick a number such that

$$a = \frac{h^2/\mu}{1-e^2} \tag{B-3}$$

where

$$h = \frac{r \times m\dot{r}}{m} = r \times \dot{r}$$

is the angular momentum per unit mass and

$$\mu = GM_E$$

in which G is the universal gravitational constant, and M_E is the mass of the earth.

Since $0 < e < 1$ for an ellipse, $e^2 < 1$. Thus, the RHS of equation (B-3) assures $a > 0$ because $h^2 > 0$ and $\mu > 0$.

Substituting (B-3) into (B-2), we have

$$r(\theta) = \frac{h^2/\mu}{1 + e\cos\theta}$$

which is identical with (22-71).

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